

Boundary State from Ellwood Invariants

Matěj Kudrna^{(a,b)1}, Carlo Maccaferri^{(a,c)2}, Martin Schnabl^{(a)3}

^(a)*Institute of Physics of the ASCR, v.v.i.
Na Slovance 2, 182 21 Prague 8, Czech Republic*

^(b)*Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles
University, V Holešovičkách 2, 180 00 Prague 8, Czech Republic*

^(c)*Dipartimento di Fisica, Università di Torino and INFN Sezione di Torino
Via Pietro Giuria 1, I-10125 Torino, Italy*

Abstract

Boundary states are given by appropriate linear combinations of Ishibashi states. Starting from any OSFT solution and assuming Ellwood conjecture we show that every coefficient of such a linear combination is given by an Ellwood invariant, computed in a slightly modified theory where it does not trivially vanish by the on-shell condition. Unlike the previous construction of Kiermaier, Okawa and Zwiebach, ours is linear in the string field, it is manifestly gauge invariant and it is also suitable for solutions known only numerically. The correct boundary state is readily reproduced in the case of known analytic solutions and, as an example, we compute the energy momentum tensor of the rolling tachyon from the generalized invariants of the corresponding solution. We also compute the energy density profile of Siegel-gauge multiple lump solutions and show that, as the level increases, it correctly approaches a sum of delta functions. This provides a gauge invariant way of computing the separations between the lower dimensional D-branes.

¹Email: matej.kudrna at email.cz

²Email: maccafer at gmail.com

³Email: schnabl.martin at gmail.com

Contents

1	Introduction	2
2	Boundary state from Ellwood invariants	5
2.1	Generalizing the Ellwood invariant	6
2.2	Ellwood invariants and Ishibashi states	9
2.3	Ellwood invariants and boundary primaries	12
3	Analytic solutions: Rolling tachyon	15
4	Numerical solutions: Lumps in Siegel gauge	22
4.1	Moeller–Sen–Zwiebach lump at $R = \sqrt{3}$	22
4.2	Double lumps at $R = 2\sqrt{3}$	28
5	Conclusions	34
A	BCFT^(aux)	36
B	Conservation laws for the Ellwood invariant	37
B.1	Review of conservation laws of the identity string field	38
B.2	Virasoro conservation laws	38
B.3	Oscillator conservation laws	41
B.4	Ghost conservation laws	42
C	General properties of the boundary state	44
C.1	Proof of matter ghost factorization	44
C.2	Normalization of the ghost boundary state	46
D	Some more lumps	51

1 Introduction

In attempts to explore the landscape of open string field theory [1]¹ either by analytic or numerical means, one faces the problem of a physical identification of the solutions to the equation of motion. They are believed to be in one-to-one correspondence with allowed boundary states for given bulk CFT, but so far we have had only limited tools to identify the respective boundary state. In [4] a geometric construction of the OSFT boundary state

¹For recent reviews see e.g. [2, 3].

was given, in principle, for any classical solution. However, due to the non-linearity of the construction, it is not known how to explicitly perform computations in more general cases, for example, for the important class of Siegel-gauge level truncated solutions. Moreover the OSFT boundary state of [4] is not guaranteed to be gauge invariant and the BCFT boundary state is recovered only up to possible BRST exact terms (which are however absent in the explicit examples of wedge-based analytic solutions). In this work we present a remarkably simple method to construct explicitly, in a gauge invariant way, the BCFT boundary state from any solution. The main advantage of the method is that, while it easily gives the expected results for known analytic solutions, it also works reasonably well for solutions known only numerically. The key ingredient of our construction is the widely believed, but as yet unproven Ellwood's conjecture [5, 4], which can be simply re-stated as

$$\langle \mathcal{V}_{cl} | c_0^- | B_\Psi \rangle = -4\pi i \langle I | \mathcal{V}_{cl}(i) | \Psi - \Psi_{TV} \rangle, \quad (1.1)$$

where \mathcal{V}_{cl} is an on-shell closed string vertex operator of ghost number two, Ψ is a solution of the OSFT, $|B_\Psi\rangle$ is the corresponding boundary state and finally Ψ_{TV} is the tachyon vacuum (in any gauge or form). Note that the left hand side is evaluated using the closed string inner product, while the right hand is evaluated using the open string inner product.

This equation, however, constrains only the tiny on-shell part of the boundary state. For example, for spatially constant string fields the only non-trivial component of the corresponding boundary state which can be computed this way is just the zero momentum massless closed string mode with vertex operator of the form $\xi_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu$. A possible way around this problem has been in fact hinted already in [6], before the Ellwood's conjecture had been formulated. The trick is essentially to assume the existence of some space-time direction with Dirichlet boundary condition. The vertex operator can then be taken to have arbitrary momentum dependence in the directions we are interested in. To put the whole operator on-shell, we adjust the momentum of the closed string vertex operator in the extra direction with Dirichlet boundary condition. Due to the Dirichlet condition the invariant will not vanish trivially. In this way we can uniquely extract the overlap between the boundary state and closed string matter primaries. The goal of this paper is to make this idea more precise and to illustrate it on the examples of the analytic rolling tachyon and numerical lump solutions describing lower dimensional D-branes.

In the matter CFT, the knowledge of the inner product of the boundary state with any primary state is sufficient to determine the complete boundary state with the help of the Virasoro gluing conditions $(L_n - \bar{L}_{-n})|B\rangle = 0$. These conditions are solved in full generality by the conformal Ishibashi states [7], and the Ellwood invariants determine the exact linear combination. In principle OSFT thus solves the outstanding unsolved problem of boundary conformal field theory: determine the set of all boundary conditions consis-

tent with conformal symmetry in a given CFT. In the string theory language this is the problem of classification of all allowed D-branes in a given background. The coefficients of the Ishibashi states must satisfy lots of constraints, either from modular invariance or the so called sewing conditions. A lot of progress has been achieved in CFT attempting to solve these constraints, but much more remains. String field theory solutions, on the other hand, should provide automatically a solution to all these constraints.

Let us describe the computation of the primary components of the boundary state in a little bit more detail. The state space of open string field theory is given by the Hilbert space of the boundary conformal field theory BCFT_0 . Any such element in the ghost number one sector can be written as [8, 9]

$$\Psi = \sum_j \sum_{\substack{I = \{n_1, n_2, \dots\} \\ J = \{m_1, m_2, \dots\}}} a_{IJ}^j L_{-I}^{\text{matter}} |\mathcal{V}_j\rangle \otimes L_{-J}^{\text{ghost}} c_1 |0\rangle, \quad (1.2)$$

where the index j runs over all matter primaries that are 'turned on', while the multi-indices I and J give its descendants. The tachyon vacuum does not turn on any primary other than the identity operator, while for example for the lumps an infinite number of e^{ikx} primaries is required.

Given any solution of string field theory built upon BCFT_0 , one can uniquely associate to it a solution built upon $\text{BCFT}_0 \otimes \text{BCFT}_{\text{aux}}$, where BCFT_{aux} is an auxiliary (non-unitary) BCFT of central charge $c = 0$. It is enough to change all L^{matter} 's into $L^{\text{matter}} + L^{\text{aux}}$. Let us further assume that BCFT_{aux} contains bulk primary operators of dimension $(1-h, 1-h)$ with non-vanishing disk 1-point function for every weight (h, h) primary in the matter part of BCFT_0 . One universal option is to choose a BCFT of a free boson (let us call it Y) with Dirichlet boundary condition and consider generically non-normalizable operators $e^{2i\sqrt{1-h}Y}$. To ensure zero central charge of BCFT_{aux} , one should tensor this free boson theory with a non-unitary theory of negative central charge, for example a $c = -1$ linear dilaton theory, and supplement the closed string insertion with the appropriate weight $(0,0)$ primary, w , to soak up the background charge.²

Since BCFT_0 and BCFT_{aux} are completely decoupled, and the solution does not turn on any BCFT_{aux} primaries other than the identity, the boundary conditions of BCFT_{aux} cannot be changed by the solution. The boundary state for the OSFT solution will thus be

$$|B_\Psi\rangle^{\text{CFT}_0 \otimes \text{CFT}_{\text{aux}}} = |B_\Psi\rangle^{\text{CFT}_0} \otimes |B_0\rangle^{\text{CFT}_{\text{aux}}}. \quad (1.3)$$

²In most cases, however, such a construction is not necessary. It is enough to assume the existence of a space-time direction along which nothing happens, and change its boundary condition to Dirichlet, if it is not Dirichlet to start with.

Assuming the boundary state factorizes in ghost and matter

$$|B_\Psi\rangle^{\text{CFT}_0} = |B_\Psi\rangle^{\text{CFT}_0^{\text{matter}}} \otimes |B_{gh}\rangle, \quad (1.4)$$

and decomposing $|B_\Psi\rangle^{\text{CFT}_0^{\text{matter}}}$ into the basis of Ishibashi states

$$|B_\Psi\rangle^{\text{CFT}_0^{\text{matter}}} = \sum_{\alpha} n_{\Psi}^{\alpha} ||V_{\alpha}\rangle\rangle \quad (1.5)$$

in the (product of left and right) Verma module of the matter primary operator V_{α} , we can determine all the coefficients from the knowledge of the generalized Ellwood invariants

$$n_{\Psi}^{\alpha} = 2\pi i \langle I | \mathcal{V}^{\alpha}(i) | \Psi - \Psi_{\text{TV}} \rangle^{\text{BCFT}_0 \otimes \text{BCFT}_{\text{aux}}}, \quad (1.6)$$

where

$$\mathcal{V}^{\alpha} = c\bar{c}V^{\alpha} e^{2i\sqrt{1-h_{\alpha}}Y} w$$

and V^{α} form a dual basis in the matter part of CFT_0 , i.e.

$$\langle V^{\alpha} | V_{\beta} \rangle = \delta_{\beta}^{\alpha}.$$

This is our main result.

The paper is organized as follows. In Sec. 2 we describe our construction of the boundary state in more detail. In Sec. 3 we derive the boundary state for analytic solutions, taking the rolling tachyon as a representative example. In Sec. 4 we apply our construction to single and double lump solutions in open string field theory. We end with some conclusions and future perspectives. Appendix A contains an example of the auxiliary $c = 0$ BCFT which we use to generalize the Ellwood invariants, in appendix B we derive a set of conservation laws for the Ellwood invariant which are a very efficient tools, especially in numerical computations. In appendix C we discuss in some detail universal properties of the boundary state in bosonic string theory. Finally, appendix D contains numerical results about different Siegel gauge lump solutions other than those analyzed in Sec. 4.

2 Boundary state from Ellwood invariants

In this section we construct a boundary state from a given OSFT solution in two steps. First, we generalize Ellwood conjecture in order to be able to use generic matter primaries in the Ellwood invariant. Then, we show that a generic boundary state describing conformal boundary conditions in a total matter/ghost BCFT is necessarily matter-ghost factorized, and use the Virasoro gluing condition of the matter sector to fix the non-primary part of the matter boundary state. Only the first part of our construction resides in OSFT.

2.1 Generalizing the Ellwood invariant

Let BCFT_0 be the reference boundary CFT on which we define OSFT. Let Ψ be a solution describing another BCFT_Ψ . Then Ellwood conjectured that [5, 4]

$$-4\pi i \langle I|\mathcal{V}(i, -i)|\Psi \rangle \equiv -4\pi i \langle E[\mathcal{V}]|\Psi \rangle = \langle \mathcal{V}|c_0^-|B_\Psi \rangle - \langle \mathcal{V}|c_0^-|B_0 \rangle. \quad (2.1)$$

Here

$$\mathcal{V} = c\bar{c}V^{\text{matter}}, \quad \langle E[\mathcal{V}] \equiv \langle I|\mathcal{V}(i, -i).$$

Because \mathcal{V} is inserted at a conical singularity (the midpoint of the identity string field) the quantity $\langle I|\mathcal{V}(i, -i)|\Psi \rangle$ is only meaningful when \mathcal{V} is a weight zero primary. Luckily all the $gh = 2$ closed string cohomology is contained in states of this form and thus (2.1) can be used to define the on-shell part of the boundary state $|B_\Psi\rangle$. But this is clearly not enough to completely define the boundary state.

This is the well-known limitation of Ellwood invariants: most of the closed string tadpoles vanish by momentum conservation when the closed string is on-shell. This limitation lead the authors of [4] to the construction of a family of Wilson-loop-like maps from the classical solution Ψ to ghost-number 3 level-matched and BRST invariant closed string states which are conjectured to be BRST-equivalent to the boundary state. In particular one can probe them with off-shell closed string states. Assuming Ellwood conjecture (or alternatively assuming background independence of a version of open-closed string field theory, [4]), the BRST equivalence to the BCFT boundary state can be established. The construction is completely done in the open string star algebra and its intrinsic non-linearity can give nontrivial checks on the regularity of proposed OSFT classical solutions, [10]. But there is a simple shortcut to get *precisely* the BCFT-boundary state described by the classical solution Ψ . Suppose we are dealing with a solution which does not depend on a target space direction, say Y . This means that the Y dependence of the solution can be taken to be universal, depending only on Virasoros of the Y -CFT. Then the solution will remain a solution if we change the boundary conditions of the Y -BCFT to be Dirichlet $Y(0, \pi) = 0$. A generic closed string vertex operator of the form $c\bar{c}V^{(h,h)}$ (where $V^{(h,h)}$ is a bulk (h, h) matter primary not depending on the Y direction) can be formally put on-shell by going to a complexified mass shell: this can be done by multiplying $c\bar{c}V^{(h,h)}$ with $e^{2i\sqrt{1-h}Y}$ which has weight $(1-h, 1-h)$. For $h > 1$ (typical case) this is a negative weight primary which is in general not normalizable due to the divergent zero mode integration in the world-sheet path integral. But this is not a problem with Dirichlet boundary conditions, as the zero mode path integral will be localized to $y = 0$. Moreover, since disk one point functions of bulk exponential operators are non-vanishing with Dirichlet boundary conditions, the corresponding modified tadpoles will also be generically nonzero.

This example suggests that, whenever we have a solution with an explicit dependence on matter Virasoro operators³ we can consider adding an auxiliary BCFT^{aux} sector of total $c = 0$ to the basis states of the original BCFT_0 . Then we automatically have a lifted solution

$$\Psi \rightarrow \tilde{\Psi} \equiv \Psi|_{L^{\text{matter}} \rightarrow L^{\text{matter}} + L^{\text{aux}}}, \quad (2.2)$$

in a lifted OSFT defined on

$$\text{BCFT}'_0 \equiv \text{BCFT}_0 \otimes \text{BCFT}^{\text{aux}}$$

with a lifted BRST charge

$$Q \rightarrow \tilde{Q} \equiv \sum_n : c_{-n} \left(L_n^{\text{matter}} + L_n^{\text{aux}} + \frac{1}{2} L_n^{\text{ghost}} \right) : \quad (2.3)$$

Now, given any CFT bulk primary of the form

$$\mathcal{V}^\alpha(z, \bar{z}) = c\bar{c}V^\alpha(z, \bar{z}), \quad (2.4)$$

where V^α is a purely matter primary of weight (h_α, h_α) , we can consider a formal bulk primary in CFT^{aux} , $w^\alpha(z, \bar{z})$ of weight $(1 - h_\alpha, 1 - h_\alpha)$ with the property that

$$\langle w^\alpha(0) \rangle_{\text{disk}}^{\text{BCFT}^{\text{aux}}} = 1, \quad \forall \alpha. \quad (2.5)$$

Explicitly, see appendix A, we can define BCFT^{aux} to be the tensor product of a free boson Y with Dirichlet boundary conditions ($c = 1$) and a linear dilaton φ with background charge $Q = \frac{1}{\sqrt{3}}$ with Neumann boundary conditions and $c = 1 - 6Q^2 = -1$. In this case we can systematically take

$$w^\alpha = e^{2i\sqrt{1-h_\alpha}Y} e^{\frac{2i}{\sqrt{3}}\varphi}, \quad (2.6)$$

which has weight $(1 - h_\alpha, 1 - h_\alpha)$ and satisfies (2.5), thanks to the Dirichlet conditions for Y and the saturation of the background charge on the disk. Notice that, for $h_\alpha > 1$, w^α is not normalizable in the auxiliary closed string Hilbert space, but still it has a well defined one-point function on the disk. Other choices of BCFT^{aux} are clearly possible.

For OSFT purposes the closed string insertion

$$\tilde{\mathcal{V}}^\alpha \equiv c\bar{c}V^\alpha \otimes w^\alpha$$

will be a total $(0, 0)$ bulk primary (in fact a formal, not normalizable, element of the \tilde{Q} closed string cohomology). Thus, *assuming Ellwood conjecture*, the Ellwood invariant will

³This is the case for all known analytic solutions. For numerical solutions we can always, level by level, express the solution in the form (1.2).

compute the difference in the tadpoles between the two BCFT's related by the classical solution. But since the solution $\tilde{\Psi}$ does not switch on any primaries in BCFT^{aux} , the generalized Ellwood invariant will be proportional to the disk one-point function of w^α

$$\begin{aligned}
& -4\pi i \langle E[\tilde{\mathcal{V}}^\alpha] | \tilde{\Psi} \rangle^{\text{BCFT}'_0} \\
&= \langle \tilde{\mathcal{V}}^\alpha | c_0^- | \tilde{B}_\Psi \rangle - \langle \tilde{\mathcal{V}}^\alpha | c_0^- | \tilde{B}_0 \rangle \\
&= \left(\langle c\bar{c}V^\alpha | \otimes \langle w^\alpha | \right) c_0^- \left(|B_\Psi\rangle \otimes |B_{\text{aux}}\rangle \right) - \left(\langle c\bar{c}V^\alpha | \otimes \langle w^\alpha | \right) c_0^- \left(|B_0\rangle \otimes |B_{\text{aux}}\rangle \right) \\
&= \left(\langle c\bar{c}V^\alpha | c_0^- | B_\Psi \rangle - \langle c\bar{c}V^\alpha | c_0^- | B_0 \rangle \right) \times \langle w^\alpha(0) \rangle_{\text{disk}}^{\text{BCFT}^{\text{aux}}} \\
&= \langle c\bar{c}V^\alpha | c_0^- | B_\Psi \rangle - \langle c\bar{c}V^\alpha | c_0^- | B_0 \rangle.
\end{aligned} \tag{2.7}$$

Notice that the auxiliary CFT disappeared from the RHS. Conveniently, we can relate the BCFT_0 -boundary state with the Ellwood invariant of the lifted tachyon vacuum, $\tilde{\Psi}_{TV}$, and we can write the ‘generalized’ Ellwood conjecture in the simple form

$$\langle c\bar{c}V^\alpha | c_0^- | B_\Psi \rangle = -4\pi i \langle E[\tilde{\mathcal{V}}^\alpha] | \tilde{\Psi} - \tilde{\Psi}_{TV} \rangle. \tag{2.8}$$

String field theory solutions related by gauge transformations should describe the same BCFT and thus the same boundary state $|B_\Psi\rangle$. Although the right hand side is manifestly invariant under the gauge transformations in the new OSFT based on BCFT'_0 , to show that it is invariant also under the gauge transformation in the original OSFT based on BCFT_0 requires a little thought. One has to show that the lifting from BCFT_0 to BCFT'_0 commutes with gauge transformations. The lift of a gauge transformed field is a lifted gauge transform of the lifted field

$$\text{Lift} \circ (\text{Gauge Transf})_\Lambda = (\text{Gauge Transf})_{\text{Lift}(\Lambda)} \circ \text{Lift}. \tag{2.9}$$

To see this, note that lifting $Q\Lambda$ one gets the same thing as acting with \tilde{Q} on lifted Λ . Similarly lifting the star product $A * B$ one finds the star product of lifted fields. This is most easily seen by decomposing both fields in the basis of matter Virasoro descendants plus ghost oscillators and using the conservation laws to compute the star product⁴. Application of the conservation laws in the lifted theory gives the same answer, since the commutation rules and operator weights are not changed by the lift.

⁴Since gauge parameters have zero ghost number they can be made of ghost non-primaries which are not descendants, like the ghost current. In this case it is convenient to span the ghost sector with oscillators and use the corresponding conservation laws.

2.2 Ellwood invariants and Ishibashi states

Using OSFT and Ellwood conjecture, we are thus in a situation where we know the overlap

$$\langle B_\Psi | c_0^- | \mathcal{V} \rangle,$$

where

$$|\mathcal{V}\rangle = V^{(h,h)}(0) c_1 \bar{c}_1 |0\rangle_{SL(2,C)}, \quad (2.10)$$

and $V^{(h,h)}(z, \bar{z})$ is a weight (h, h) bulk primary in the matter sector. It follows that

$$L_{n \geq 1}^{\text{matter}} |\mathcal{V}\rangle = \bar{L}_{n \geq 1}^{\text{matter}} |\mathcal{V}\rangle = 0 \quad (2.11)$$

$$c_{n \geq 1} |\mathcal{V}\rangle = \bar{c}_{n \geq 1} |\mathcal{V}\rangle = 0 \quad (2.12)$$

$$b_{n \geq 0} |\mathcal{V}\rangle = \bar{b}_{n \geq 0} |\mathcal{V}\rangle = 0. \quad (2.13)$$

Clearly, the closed string states $c\bar{c}V^{(h,h)}$ do not span the whole set of off-shell closed string fields. However, any generic off-shell closed string state can be obtained acting with ghost oscillators and matter Virasoros on the ground states given by the $|\mathcal{V}\rangle$'s. This choice of basis is quite convenient from the OSFT point of view, since the states $|\mathcal{V}\rangle$ directly enter in the Ellwood invariant. But it is also very convenient from the closed string point of view: the knowledge of the overlap $\langle \mathcal{V} | c_0^- | B_\Psi \rangle$ is just enough to define *all* overlaps with the boundary state.

This is because the boundary state is a ghost number three closed string state which describes conformal boundary conditions in the total matter/ghost Hilbert space. This is summarized by

$$b_0^- |B_\Psi\rangle = 0 \quad (2.14)$$

$$(L_n^{\text{tot}} - \bar{L}_{-n}^{\text{tot}}) |B_\Psi\rangle = 0 \quad (2.15)$$

$$(Q_{gh} - 3) |B_\Psi\rangle = 0, \quad (2.16)$$

where Q_{gh} is the total ghost number operator obeying $Q_{gh} |0\rangle_{SL(2,C)} = 0$. We show in appendix B how these conditions imply the standard gluing conditions

$$(b_n - \bar{b}_{-n}) |B_\Psi\rangle = 0 \quad (2.17)$$

$$(c_n + \bar{c}_{-n}) |B_\Psi\rangle = 0 \quad (2.18)$$

$$(L_n^{\text{matter}} - \bar{L}_{-n}^{\text{matter}}) |B_\Psi\rangle = 0 \quad (2.19)$$

$$(Q + \bar{Q}) |B_\Psi\rangle = 0, \quad (2.20)$$

These gluing conditions allow to trade raising operators acting on the closed string state $|\mathcal{V}\rangle$ with lowering operators which will vanish upon acting on $|\mathcal{V}\rangle$. Thus any overlap

of a closed string state $|W\rangle$ build by acting raising operators on $|\mathcal{V}\rangle$ will be proportional to the corresponding overlap of the boundary state with $|\mathcal{V}\rangle$ itself, the constant of proportionality being a number which can be easily computed using the matter Virasoro algebra and the b, c oscillator algebra. *Thus, up to automatic operations, the boundary state for a solution Ψ is completely encoded in the (generalized) Ellwood invariants.*

We can beautifully formulate these observations in terms of the so-called Ishibashi states. Let $\{V_\alpha\}$ be the space of non-singular *spinless* bulk primaries of weight (h_α, h_α) in the matter CFT.⁵

$$(L_0 - \bar{L}_0)|V_\alpha\rangle = (h_\alpha - h_\alpha)|V_\alpha\rangle = 0 \quad (2.21)$$

$$L_n|V_\alpha\rangle = \bar{L}_n|V_\alpha\rangle = 0, \quad n \geq 1. \quad (2.22)$$

Let's define a BPZ-dual basis of primaries $\{V^\beta\}$ such that

$$\langle V^\alpha | V_\beta \rangle = \delta^\alpha_\beta. \quad (2.23)$$

This is possible once singular (null) states have been projected out. To any spinless vertex operator V_α we can associate the corresponding conformal Ishibashi state which (up to normalization) is the unique state $||V_\alpha\rangle\rangle$ in the Virasoro Verma module of V_α satisfying the Virasoro gluing conditions

$$(L_n - \bar{L}_{-n})||V_\alpha\rangle\rangle = 0. \quad (2.24)$$

The explicit form of the Ishibashi state $||V_\alpha\rangle\rangle$ to any desired level can be found easily by solving the gluing conditions in the Verma module of V_α level by level, and one finds in the absence of null states

$$||V_\alpha\rangle\rangle = \left[1 + \frac{1}{2h_\alpha} L_{-1} \bar{L}_{-1} + B(h_\alpha, c) \left(2(1 + 2h_\alpha) L_{-2} \bar{L}_{-2} - 3(L_{-2} \bar{L}_{-1}^2 + L_{-1}^2 \bar{L}_{-2}) + \frac{8h_\alpha + c}{4h_\alpha} L_{-1}^2 \bar{L}_{-1}^2 \right) + \dots \right] |V_\alpha\rangle, \quad (2.25)$$

$$B(h_\alpha, c) = \frac{1}{2h_\alpha(8h_\alpha - 5) + c(2h_\alpha + 1)}. \quad (2.26)$$

Had there been a null state at some level, the coefficients in this expression at that level would be divergent. For example the level 2 null state appears exactly for those values of h and c for which $B(h, c)$ diverges. In such a case one should exclude the null states from the Verma module. Solving then the gluing conditions with null states projected

⁵In CFTs on noncompact target spaces α will in general be a continuous variable, like the momentum.

out gives analogous expression to (2.25) but with finite coefficients. A simple closed form of the solution to the gluing condition for the general case has been found by Ishibashi [7]. One can write it as

$$||V_\alpha\rangle\rangle = \sum_n |n, \alpha\rangle \otimes \overline{|n, \alpha\rangle}, \quad (2.27)$$

where the sum runs over orthonormal basis of states in the irreducible representation of the chiral Virasoro algebra built over the primary V_α . We have also assumed that the closed string primary V_α can be decomposed into product of holomorphic and antiholomorphic parts. Equivalently we can write [11]

$$||V_\alpha\rangle\rangle = \sum_{IJ} M^{IJ}(h_\alpha) L_{-I} \bar{L}_{-J} |V_\alpha\rangle, \quad (2.28)$$

where the indices I, J label the non-degenerate descendants in the conformal family of V_α and $M^{IJ}(h_\alpha)$ is defined as the inverse of the real symmetric matrix

$$M_{IJ}(h_\alpha) = \langle V^\alpha | L_I L_{-J} | V_\alpha \rangle = \langle V^\alpha | \bar{L}_I \bar{L}_{-J} | V_\alpha \rangle. \quad (2.29)$$

The normalization has been chosen such that

$$\langle V^\alpha ||V_\beta\rangle\rangle = \langle V^\alpha | V_\beta \rangle = \delta_\beta^\alpha. \quad (2.30)$$

Any boundary state $|B_*\rangle$ in the matter CFT is therefore written as

$$|B_*\rangle = \sum_\alpha n_*^\alpha ||V_\alpha\rangle\rangle. \quad (2.31)$$

If we want to *define* BCFT_{*} through its boundary state, the n_*^α must be precisely chosen in order to satisfy Cardy conditions (modular invariance) and sewing conditions (factorization of bulk n-point functions in open and closed string channels), see e.g. [12]. If the boundary state $|B_*\rangle$ is known, we can easily get n_*^α by

$$n_*^\alpha = \langle V^\alpha | B_* \rangle. \quad (2.32)$$

In the OSFT approach to BCFT, instead of searching for linear combinations of Ishibashi states obeying nontrivial consistency conditions, we search for solutions to the equation of motion. If OSFT is a consistent theory we expect such classical solutions to automatically describe consistent boundary conditions.

With the above premises, our proposal can be compactly written as

$$|B_\Psi\rangle = \sum_{\alpha} n_{\Psi}^{\alpha} ||V_{\alpha}\rangle\rangle \otimes |B_{gh}\rangle, \quad (2.33)$$

$$n_{\Psi}^{\alpha} \equiv 2\pi i \left\langle E[\tilde{\mathcal{V}}^{\alpha}] | \tilde{\Psi} - \tilde{\Psi}_{TV} \right\rangle, \quad (2.34)$$

where we used, see also appendix C

$$\langle 0 | c_{-1} \bar{c}_{-1} c_0^- | B_{gh} \rangle = \langle (c_0 - \bar{c}_0) c \bar{c}(0) \rangle_{\text{disk}}^{\text{ghost}} = -2 \quad (2.35)$$

$$\mathcal{V}^{\alpha} = c \bar{c} V^{\alpha} \quad (2.36)$$

$$\tilde{\mathcal{V}}^{\alpha} = \mathcal{V}^{\alpha} \otimes w^{\alpha}, \quad (2.37)$$

and Ψ_{TV} is any OSFT solution for the tachyon vacuum whose contribution replaces the corresponding contribution from the BCFT₀ boundary state.

2.3 Ellwood invariants and boundary primaries

It is useful and interesting to elucidate the relation between the primary boundary fields that are ‘switched on’ in an OSFT classical solution and the boundary state which is associated to the solution via the Ellwood conjecture. To this end we consider the solution expressed as

$$\Psi - \Psi_{TV} = \sum_j \sum_{I,J} a_{IJ}^j L_{-I}^{\text{matter}} |\phi_j\rangle \otimes L_{-J}^{\text{ghost}} c_1 |0\rangle, \quad (2.38)$$

where I, J are multi-indices of the form

$$\{n_k, \dots, n_1\}, \quad n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1 \quad (2.39)$$

and $L_{-I}^{\text{matter}}, L_{-J}^{\text{ghost}}$ stand for the corresponding product of negatively moded Virasoros

$$L_{-X} \equiv L_{-n_k} \dots L_{-n_1}, \quad (2.40)$$

acting respectively on the matter primary $|\phi_j\rangle$ and the unique ghost primary at ghost number one $c_1|0\rangle$.

The corresponding solution in BCFT₀'=BCFT₀⊗BCFT^{aux} will be given by

$$\tilde{\Psi} - \tilde{\Psi}_{TV} = \sum_j \sum_{I,J} a_{IJ}^j \tilde{L}_{-I}^{\text{matter}} |\tilde{\phi}_j\rangle \otimes L_{-J}^{\text{ghost}} c_1 |0\rangle, \quad (2.41)$$

where

$$|\tilde{\phi}_j\rangle = |\phi_j\rangle \otimes |0\rangle^{\text{aux}} \quad (2.42)$$

$$\tilde{L}_n^{\text{matter}} = L_n^{\text{matter}} + L_n^{\text{aux}}, \quad (2.43)$$

Consider now a generalized Ellwood invariant corresponding to a closed string field

$$\tilde{\mathcal{V}}^\beta = \mathcal{V}^\beta \otimes w^\beta = c\bar{c}V^\beta \otimes w^\beta, \quad (2.44)$$

where w^β is a weight $(1 - h_\beta)$ primary in the $c = 0$ BCFT^{aux} with unit one-point function on the disk, see Appendix C.

In computing $\langle E[\tilde{\mathcal{V}}^\beta] | \tilde{\Psi} - \tilde{\Psi}_{TV} \rangle$ it is useful to consider the conservation laws of the anomalous derivations

$$K_n = L_n - (-1)^n L_{-n}, \quad (2.45)$$

in matter and ghost sectors separately, which read, [6], (see appendix B for a simple derivation)

$$\begin{aligned} \langle E[\tilde{\mathcal{V}}^\beta] | \tilde{K}_{2n+1}^{(m)} &= \langle E[\tilde{\mathcal{V}}^\beta] | K_{2n+1}^{(gh)} = 0 \\ \langle E[\tilde{\mathcal{V}}^\beta] | \tilde{K}_{2n}^{(m)} &= -3(-1)^n n \langle E[\tilde{\mathcal{V}}^\beta] | \\ \langle E[\tilde{\mathcal{V}}^\beta] | K_{2n}^{(gh)} &= 3(-1)^n n \langle E[\tilde{\mathcal{V}}^\beta] |. \end{aligned} \quad (2.46)$$

Thanks to these conservation laws one can get rid of all the Virasoros in the solution, level by level. It follows that⁶

$$\langle E[\tilde{\mathcal{V}}^\beta] | \tilde{\Psi} - \tilde{\Psi}_{TV} \rangle = \sum_j A_\Psi^j \langle E[\tilde{\mathcal{V}}^\beta] | c\phi_j \rangle^{\text{BCFT}'_0}. \quad (2.47)$$

Here A_Ψ^j is a *gauge invariant* linear combination of the coefficients a_{IJ}^α which is determined by the conservation laws (2.46)⁷

$$A_\Psi^j = \sum_{I,J} K_{IJ}^{(h_j)} a_{IJ}^j. \quad (2.48)$$

⁶Since we are computing an Ellwood invariant, the tachyon vacuum solution can be traded for the simple string field

$$\tilde{\Psi}_{TV} \rightarrow \frac{2}{\pi} c_1 |0\rangle.$$

Thus, in the Ellwood invariant, the difference between Ψ and $(\Psi - \Psi_{TV})$ only appears in a universal shift in the coefficient of the zero momentum tachyon.

⁷Sometimes it is more convenient to choose other basis to numerically determine the solution. For example it is customary to span the ghost Hilbert space with oscillators. Use of matter oscillators is also convenient especially in the zero momentum sector where higher level primaries appear. In this case other kind of conservation laws are needed, see appendix B. The final result after use of conservation laws is independent, level by level, on the chosen basis.

Notice that the $K^{(h_j)}$'s only depend on the weight of ϕ_j , and are otherwise completely universal.

Computing the Ellwood invariant we find

$$\begin{aligned}
-4\pi i \left\langle E[\tilde{\mathcal{V}}^\beta] \Big| c\phi_j \right\rangle^{\text{BCFT}'_0} &= -4\pi i \left(\frac{2}{\pi} \right)^{h_j-1} \langle \tilde{\mathcal{V}}^\beta(i\infty) c\phi_j(0) \rangle_{C_1}^{\text{BCFT}'_0} \\
&= -4\pi i \left(\frac{2}{\pi} \right)^{h_j-1} \frac{1}{2\pi i} |2\pi i|^{h_j} \langle \tilde{\mathcal{V}}^\beta(0) c\phi_j(1) \rangle_{\text{disk}}^{\text{BCFT}'_0} \\
&= -4^{h_j} \pi \langle \tilde{\mathcal{V}}^\beta(0) c\phi_j(1) \rangle_{\text{disk}}^{\text{BCFT}'_0} \\
&= \pi 4^{h_j} B_j^\beta,
\end{aligned} \tag{2.49}$$

where

$$\langle c\bar{c}(0)c(1) \rangle_{\text{disk}}^{\text{ghost}} = -1 \langle w^\beta(0) \rangle_{\text{disk}}^{\text{BCFT}_{\text{aux}}} = 1 \tag{2.50}$$

$$\langle V^\beta(0)\phi_j(1) \rangle_{\text{disk}}^{\text{BCFT}_0^{\text{matter}}} = B_j^\beta. \tag{2.51}$$

Notice that B_j^β are the basic bulk-boundary two point functions of the matter part of BCFT_0 . Using this we can express the coefficient in front of the Ishibashi state for the boundary state described by Ψ in terms of the bare bones

$$n_\Psi^\beta = -\frac{\pi}{2} \sum_j 4^{h_j} A_\Psi^j \langle V^\beta(0)\phi_j(1) \rangle_{\text{disk}}^{\text{BCFT}_0^{\text{matter}}}. \tag{2.52}$$

As a consistency check we derive the same formula directly on the upper half plane

$$\begin{aligned}
-4\pi i \left\langle E[\tilde{\mathcal{V}}^\beta] \Big| c\phi_j \right\rangle^{\text{BCFT}'_0} &= -4\pi i \left\langle \tilde{V}^\beta(i, -i) f_I \circ c\phi_j(0) \right\rangle_{\text{UHP}}^{\text{BCFT}'_0} \\
&= -4\pi i (f'_I(0))^{h_j-1} \left\langle \tilde{V}^\beta(i, -i) c\phi_j(0) \right\rangle_{\text{UHP}}^{\text{BCFT}'_0}
\end{aligned} \tag{2.53}$$

$$f_I(z) = \frac{2z}{1-z^2}. \tag{2.54}$$

Factorizing the correlator in ghost, matter and the auxiliary sectors

$$\left\langle \tilde{V}^\beta(i, -i) c\phi_j(0) \right\rangle_{\text{UHP}}^{\text{BCFT}'_0} = \langle c(i)c(-i)c(0) \rangle_{\text{UHP}} \langle V^\beta(i, -i)\phi_j(0) \rangle_{\text{UHP}} \langle w^\beta(i, -i) \rangle_{\text{UHP}}, \tag{2.55}$$

we find

$$\langle c(i)c(-i)c(0) \rangle_{\text{UHP}} = 2i \tag{2.56}$$

$$\langle w^\beta(i, -i) \rangle_{\text{UHP}} = 4^{h_\beta-1} \langle w^\beta(0, 0) \rangle_{\text{disk}} = 4^{h_\beta-1}. \tag{2.57}$$

where use of (2.5) and (2.35) has been made. In total we thus get

$$\boxed{n_{\Psi}^{\beta} = -\frac{\pi}{2} \sum_j 2^{h_j} 4^{h_{\beta}} A_{\Psi}^j \langle V^{\beta}(i) \phi_j(0) \rangle_{\text{UHP}}^{\text{BCFT}_0^{\text{matter}}}}. \quad (2.58)$$

Consistently we find

$$\begin{aligned} \langle V^{\beta}(i) \phi_j(0) \rangle_{\text{UHP}} &= 2^{h_j} 4^{-h_{\beta}} \langle V^{\beta}(0) \phi_j(1) \rangle_{\text{Disk}} \\ &= |f'(0)|^{h_j} |f'(i)|^{2h_{\beta}} \langle V^{\beta}(0) \phi_j(1) \rangle_{\text{disk}} \\ f(z) &= \frac{1+iz}{1-iz}. \end{aligned}$$

Notice that the formulas (2.52) or (2.58) explicitly express the boundary state in terms of the BCFT_0 primaries that are switched on in the solution Ψ . As we will see later on, this is very useful to identify the boundary conditions described by *any* numerical solution which, level by level, is always of the form (2.38) or equivalent forms.

3 Analytic solutions: Rolling tachyon

The aim of this section is to illustrate our construction in an explicit case where Ellwood conjecture has been verified, and all OSFT computations have been done already. We select the simplest well-defined OSFT solutions corresponding to marginal deformations of the initial BCFT_0 , where the marginal current has regular OPE with itself. The whole construction can be readily extended [5] to the Kiermaier-Okawa solutions [13, 14], as well as to any other example in which the Ellwood invariant has been shown to analytically compute the tadpole shift, for example [15, 16]. For definiteness we select the rolling tachyon marginal deformations generated by the marginal current $V = e^{X^0}$.

These solutions have been constructed in the \mathcal{B}_0 -gauge in [17, 18] and extended to more general gauges in [19, 20]

$$\Psi_{\lambda} = Fc \frac{B}{1 + \lambda e^{X^0} \frac{1-F^2}{K}} \lambda c e^{X^0} F, \quad (3.1)$$

where $F = F(K)^8$ and K, B, c are the familiar string fields [21, 22, 23, 24, 25].

Given an on-shell weight-zero primary closed string state $\mathcal{V} = c\bar{c}V^{(1,1)}$, the Ellwood invariant for this class of solutions has been computed in three different ways [26, 27,

⁸We assume the conditions $F(0) = 1$, $F'(0) < 0$ and $F(\infty) = 0$.

28], and the result (conveniently subtracted from the BCFT_0 contribution, given by the tachyon-vacuum invariant) is

$$\begin{aligned}\langle E[\mathcal{V}]|\Psi - \Psi_{TV}\rangle &= -\left\langle e^{-\lambda \int_0^1 ds e^{X^0}(s)} \mathcal{V}(i\infty)c(0) \right\rangle_{C_1}^{\text{BCFT}_0} \\ &= -\frac{1}{2\pi i} \left\langle e^{-\lambda \int_0^{2\pi} d\theta e^{X^0}(e^{i\theta})} \mathcal{V}(0)c(1) \right\rangle_{\text{disk}}^{\text{BCFT}_0}.\end{aligned}\quad (3.2)$$

The non-trivial rearrangement of the e^{X^0} insertions in the solution into a simple boundary interaction is a general consequence of the particular form of the solution and the string field $F(K)$, [28, 29].

This closed string tadpole is in fact closely related to the proper overlap of a closed string of the form $c\bar{c}V_m$ with the boundary state of Ψ .

$$\begin{aligned}\left\langle e^{-\lambda \int_0^{2\pi} d\theta e^{X^0}(e^{i\theta})} c\bar{c}V_m^{(h,h)}(0)c(1) \right\rangle_{\text{disk}}^{\text{BCFT}_0} &= \frac{1}{2} \left\langle e^{-\lambda \int_0^{2\pi} d\theta e^{X^0}(e^{i\theta})} (c_0 - \bar{c}_0) c\bar{c}V_m^{(h,h)}(0) \right\rangle_{\text{disk}}^{\text{BCFT}_0} \\ &\equiv \frac{1}{2} \langle B_\Psi | c_0^- | c\bar{c}V_m^{(h,h)} \rangle,\end{aligned}\quad (3.3)$$

where in the last line we have used the defining expression for the boundary state.

Notice that although this relation is trivially true for any matter operator there is no gauge invariant observable in the OSFT defined on BCFT_0 (with generic boundary conditions) that could give the LHS of (3.3) for $h \neq 1$.

In the case at hand we have the luxury of an analytic solution and it is evident that the solution depends universally on all space directions and non-universally in time. Therefore, instead of tensoring an auxiliary BCFT, we can select any space direction, say $Y = X^{25}$, and assign it Dirichlet boundary conditions. Because of universality in Y , the solution remains a solution in a different OSFT (in fact it just describes the rolling tachyon on a lower dimensional brane). Let's call $\tilde{\Psi}$ such a solution

$$\tilde{\Psi} = \Psi(Y \rightarrow \text{Dirichlet}). \quad (3.4)$$

Let's call BCFT'_0 the same starting BCFT_0 of 26 free bosons and ghosts where the free boson Y has now Dirichlet boundary conditions, namely

$$\text{BCFT}_0 = \text{BCFT}_{\text{ghost}} \otimes \widehat{\text{BCFT}}_0 \otimes \text{BCFT}_Y \quad (3.5)$$

$$\text{BCFT}'_0 = \text{BCFT}_{\text{ghost}} \otimes \widehat{\text{BCFT}}_0 \otimes \text{BCFT}_Y^{\text{Dir}} \quad (3.6)$$

Consider now $\mathcal{V}^{(h)} = c\bar{c}\hat{V}^{(h,h)}$, where $\hat{V}^{(h,h)}$ is a weight h level-matched primary of $\widehat{\text{BCFT}}_0$. The state can be turned into a weight zero primary with a non-vanishing tadpole in BCFT'_0

$$\tilde{\mathcal{V}}^{(h)} = \mathcal{V}^{(h)} e^{2\sqrt{h-1}Y}. \quad (3.7)$$

Now we compute an Ellwood invariant in this slightly modified OSFT

$$\begin{aligned}
\langle E[\tilde{\mathcal{V}}^{(h)}]|\tilde{\Psi} - \tilde{\Psi}_{TV}\rangle &= -\left\langle e^{-\lambda \int_0^1 ds e^{X^0}(s)} \tilde{\mathcal{V}}^{(h)}(i\infty) c(0) \right\rangle_{C_1}^{\text{BCFT}'_0} \\
&= -\frac{1}{2\pi i} \left\langle e^{-\lambda \int_0^{2\pi} d\theta e^{X^0}(e^{i\theta})} \tilde{\mathcal{V}}^{(h)}(0) c(1) \right\rangle_{\text{disk}}^{\text{BCFT}'_0} \\
&= -\frac{1}{2\pi i} \langle c\bar{c}(0) c(1) \rangle \left\langle e^{-\lambda \int_0^{2\pi} d\theta e^{X^0}(e^{i\theta})} \hat{V}^{h,h}(0) \right\rangle_{\text{disk}}^{\widehat{\text{BCFT}}_0} \left\langle e^{2\sqrt{h-1}Y}(0) \right\rangle^{\text{BCFT}_Y^{Dir}} \\
&= -\frac{1}{4\pi i} \frac{1}{g_Y} \left\langle e^{-\lambda \int_0^{2\pi} d\theta e^{X^0}(e^{i\theta})} (c_0 - \bar{c}_0) c\bar{c} \hat{V}^{h,h}(0) \right\rangle_{\text{disk}}^{\text{BCFT}_0}, \tag{3.8}
\end{aligned}$$

where we wrote, (A.1)

$$1 = \left\langle e^{2\sqrt{h-1}Y}(0) \right\rangle_{\text{disk}}^{\text{BCFT}_Y^{Dir}}, \tag{3.9}$$

and

$$g_Y = \langle 1 \rangle_{\text{disk}}^{\text{BCFT}_Y} \tag{3.10}$$

is the disk partition function of the original BCFT_Y .

We thus found

$$\begin{aligned}
\langle c\bar{c} \hat{V}^{(h,h)} | c_0^- | B_\Psi \rangle &= -4\pi i g_Y \langle E[\tilde{\mathcal{V}}^{(h)}] | \tilde{\Psi} - \tilde{\Psi}_{TV} \rangle \\
&= \left\langle c\bar{c} \hat{V}^{h,h}(0) (c_0 - \bar{c}_0) e^{-\lambda \int_0^{2\pi} d\theta e^{X^0}(e^{i\theta})} \right\rangle_{\text{disk}}^{\text{BCFT}_0}. \tag{3.11}
\end{aligned}$$

Since the boundary state of the solution must be of the form

$$|B_\Psi\rangle = |\widehat{B}_\Psi\rangle \otimes |B_Y\rangle \otimes |B_{gh}\rangle \tag{3.12}$$

we have that

$$\langle \hat{V}^{(h,h)} | |\widehat{B}_\Psi\rangle = \left\langle e^{-\lambda \int_0^{2\pi} d\theta e^{X^0}(e^{i\theta})} \hat{V}^{h,h}(0) \right\rangle_{\text{disk}}^{\widehat{\text{BCFT}}_0}. \tag{3.13}$$

Once this is true for any level-matched primary of $\widehat{\text{BCFT}}_0$, it follows from the Virasoro gluing conditions that

$$|\widehat{B}_\Psi\rangle = e^{-\lambda \int_0^{2\pi} d\theta e^{X^0}} |\widehat{B}_0\rangle, \tag{3.14}$$

and then

$$|B_\Psi\rangle = e^{-\lambda \int_0^{2\pi} d\theta e^{X^0}} |B_0\rangle, \tag{3.15}$$

where $|B_0\rangle$ is the boundary state of BCFT_0 .

To elucidate the relation between Ishibashi states and Ellwood invariants we can compute the energy momentum tensor of the solution. Following Sen (appendix A of [30]), the energy momentum tensor can be extracted from the general form of a boundary state describing a configuration of branes in flat Minkowski spacetime.

$$|B\rangle = \int \frac{d^{26}k}{(2\pi)^{26}} [F(k) + (A_{\mu\nu}(k) + C_{\mu\nu}(k))\alpha_{-1}^{\mu}\bar{\alpha}_{-1}^{\nu} + B(k)(b_{-1}\bar{c}_{-1} + \bar{b}_{-1}c_{-1}) + \dots] c_0^+ c_1 \bar{c}_1 |0, k\rangle, \quad (3.16)$$

where

$$A_{\mu\nu} = A_{\nu\mu} \quad (3.17)$$

$$C_{\mu\nu} = -C_{\nu\mu}. \quad (3.18)$$

Using this in the linearized equation of motion of Closed String Field Theory, one finds that the source of the graviton (i.e. the energy-momentum tensor) is given by

$$T_{\mu\nu}(k) = \frac{1}{2} (A_{\mu\nu}(k) + \eta_{\mu\nu} B(k)). \quad (3.19)$$

By inspection we find that⁹

$$A^{\mu\nu}(k) = -\frac{1}{2} \langle 0, -k | c_{-1} \bar{c}_{-1} \alpha_1^{(\mu} \bar{\alpha}_1^{\nu)} c_0^- | B \rangle \quad (3.20)$$

$$B(k) = -\frac{1}{2} \langle 0, -k | c_{-1} \bar{c}_{-1} \frac{1}{2} (c_1 \bar{b}_1 + \bar{c}_1 b_1) c_0^- | B \rangle, \quad (3.21)$$

Notice that $B(k)$ is the overlap of the boundary state with the ghost dilaton

$$(c\partial^2 c + \bar{c}\bar{\partial}^2 \bar{c})e^{ik \cdot X}$$

which is not a primary field. This seems to imply that $B(k)$ cannot be computed from an Ellwood invariant. However, using the bc -gluing conditions

$$(c_1 + \bar{c}_{-1})|B\rangle = (b_1 - \bar{b}_{-1})|B\rangle = 0 \quad (3.22)$$

we find that $B(k)$ is also (minus) the overlap with the closed string tachyon, which is a primary field.

$$B(k) = \frac{1}{2} \langle 0, -k | c_{-1} \bar{c}_{-1} c_0^- | B \rangle, \quad (3.23)$$

⁹We use the normalization for the BPZ inner product

$$\langle 0, k | c_{-1} c_0 c_1 \bar{c}_{-1} \bar{c}_0 \bar{c}_1 | 0, k' \rangle = (2\pi)^{26} \delta(k + k').$$

In other words, looking at (3.16), we find, on general grounds

$$B(k) = -F(k). \quad (3.24)$$

Since we are studying a spatially homogeneous process, only the timelike component $k_0 \equiv -iq$ of the momentum enters the computation.¹⁰ We have

$$B_\Psi(q) = \frac{1}{2} \langle e^{-qX^0} | c_{-1} \bar{c}_{-1} c_0^- | B_\Psi \rangle = -2\pi i g_Y \langle E[\tilde{\mathcal{V}}_T] | \tilde{\Psi} - \tilde{\Psi}_{TV} \rangle \quad (3.25)$$

$$\tilde{\mathcal{V}}_T = c \bar{c} e^{-qX^0} e^{2\sqrt{\frac{q^2}{4}-1}Y}. \quad (3.26)$$

From the previous computation we find

$$B_\Psi(q) = -f_\lambda(q) \text{Vol}_{25}, \quad (3.27)$$

$$f_\lambda(q) \equiv \left\langle e^{-\lambda \int_0^{2\pi} d\theta e^{X^0(e^{i\theta})}} e^{-qX^0}(0) \right\rangle_{\text{disk}}^{X^0}, \quad (3.28)$$

notice that $g_Y = \text{Vol}_Y$ has been absorbed into the *space* volume Vol_{25} . Notice also that $B_\Psi(q)$ is just the coefficient of the Ishibashi state $|e^{qX^0}\rangle$ in the boundary state.

Continuing with $A_\Psi^{ij}(q)$ we have

$$A_\Psi^{ij}(q) = -\frac{1}{2} \langle e^{-qX^0} | c_{-1} \bar{c}_{-1} \alpha_1^{(i} \bar{\alpha}_1^{j)} c_0^- | B_\Psi \rangle = 2\pi i g_Y \langle E[\tilde{\mathcal{V}}^{ij}] | \tilde{\Psi} - \tilde{\Psi}_{TV} \rangle \quad (3.29)$$

$$\tilde{\mathcal{V}}^{ij} = -2c\bar{c}\partial X^{(i}\bar{\partial} X^{j)} e^{-qX^0} e^{qY}. \quad (3.30)$$

Computing the Ellwood invariant gives

$$A_\Psi^{ij}(q) = -f_\lambda(q) \delta^{ij} \text{Vol}_{25}. \quad (3.31)$$

We also trivially have

$$A_\Psi^{i0}(q) = 0. \quad (3.32)$$

It is less straightforward to compute A_Ψ^{00} . To get this contribution we have to contract the boundary state with the closed string state $W = -2 : \partial X^0 \bar{\partial} X^0 e^{-qX^0} : \text{.}$ Due to normal ordering this is not a primary field (for nonzero momentum) and we cannot directly compute this contribution from the Ellwood invariant. We have first to decompose the state in primaries and descendants, and use the Virasoro gluing conditions of the boundary state to reexpress the contribution from descendants in terms of primaries.

$$\langle e^{-qX^0} | c_{-1} \bar{c}_{-1} \alpha_1^0 \bar{\alpha}_1^0 = -\frac{2}{q^2} \langle e^{-qX^0} | c_{-1} \bar{c}_{-1} L_1^{\text{matter}} \bar{L}_1^{\text{matter}}, \quad q \neq 0. \quad (3.33)$$

¹⁰We define $|e^{qX^0}\rangle \equiv |0, ik_0\rangle$, we mimic the needed Wick rotation in time by setting $\langle e^{qX^0} | e^{q'X^0} \rangle^{(X^0)} = 2\pi\delta(q+q')$.

For nonzero momentum we thus have

$$\begin{aligned}
A_{\Psi}^{00}(q) &= -\frac{1}{2} \langle e^{-qX^0} | c_{-1} \bar{c}_{-1} \alpha_1^0 \bar{\alpha}_1^0 c_0^- | B_{\Psi} \rangle \\
&= \frac{1}{2} \frac{2}{q^2} \langle e^{-qX^0} | c_{-1} \bar{c}_{-1} L_1^{\text{matter}} \bar{L}_1^{\text{matter}} c_0^- | B_{\Psi} \rangle \\
&= \frac{1}{q^2} \langle e^{-qX^0} | c_{-1} \bar{c}_{-1} [L_1^{\text{matter}}, L_{-1}^{\text{matter}}] c_0^- | B_{\Psi} \rangle \\
&= \frac{1}{q^2} 2 \frac{q^2}{4} \langle e^{-qX^0} | c_{-1} \bar{c}_{-1} c_0^- | B_{\Psi} \rangle \\
&= B_{\Psi}(q) = -f_{\lambda}(q) \text{Vol}_{25}, \quad q \neq 0.
\end{aligned} \tag{3.34}$$

Notice that this contribution comes from the first nontrivial level of the Ishibashi state

$$||e^{qX^0}\rangle\rangle = \left(1 + \frac{1}{2h} L_{-1}^{\text{matter}} \bar{L}_{-1}^{\text{matter}} + \dots\right) |e^{qX^0}\rangle,$$

thus it is not surprising that we get the same result as if we contracted the boundary state with the closed string tachyon.

For $q = 0$, $\partial X^0 \bar{\partial} X^0 e^{-qX^0}$ is a primary and probes the Ishibashi state $||\partial X^0 \bar{\partial} X^0\rangle\rangle$. Thus for zero momentum we have

$$A^{00}(q=0) = -\frac{1}{2} \langle 0 | c_{-1} \bar{c}_{-1} \alpha_1^0 \bar{\alpha}_1^0 c_0^- | B_{\Psi} \rangle = 2\pi i \langle E[\mathcal{V}^{00}] | \Psi - \Psi_{TV} \rangle \tag{3.35}$$

$$\mathcal{V}^{00} = -2c\bar{c}\partial X^0 \bar{\partial} X^0, \tag{3.36}$$

which gives

$$\begin{aligned}
A^{00}(q=0) &= \text{Vol}_{25} \left\langle e^{-\lambda \int_0^{2\pi} d\theta e^{X^0}(e^{i\theta})} (-2) \partial X^0 \bar{\partial} X^0(0) \right\rangle_{\text{disk}}^{X^0} \\
&= \text{Vol}_{26}.
\end{aligned} \tag{3.37}$$

Notice that because of momentum conservation the boundary interaction is not giving any contribution, this would not be the case for the cosh X^0 deformation. The energy momentum tensor in q -space is thus given by

$$T_{\Psi}^{ij}(q) = \frac{1}{2} (A_{\Psi}^{ij}(q) + \delta^{ij} B(q)) = -f_{\lambda}(q) \delta^{ij} \text{Vol}_{25} \tag{3.38}$$

$$T^{i0}(q) = 0 \tag{3.39}$$

$$T^{00}(q) = \frac{1}{2} (A_{\Psi}^{00}(q) + \eta^{00} B_{\Psi}(q)) = 0, \quad q \neq 0 \tag{3.40}$$

$$T^{00}(q=0) = \frac{1}{2} (A_{\Psi}^{00}(0) + \eta^{00} B_{\Psi}(0)) = \frac{1}{2} [\text{Vol}_{26} + f_{\lambda}(0) \text{Vol}_{25}].$$

Looking at the definition of f_λ we see that

$$f_\lambda(0) = \langle e^{-\lambda \int e^{X^0}} \rangle_{\text{disk}} = \langle 1 \rangle_{\text{disk}} = \text{Vol}_{(X^0)}, \quad (3.41)$$

with our zero mode normalization we have

$$\text{Vol}_{(X^0)} = \langle 0|0 \rangle = 2\pi\delta(0), \quad (3.42)$$

in other words, we found

$$T_\Psi^{00}(q) = 2\pi\delta(q) \text{Vol}_{25}. \quad (3.43)$$

It remains to compute the disk amplitude $f_\lambda(q)$. In fact this amplitude has been computed by Larsen *et al.* in [31]. Here our approach is seemingly different but equivalent: instead of getting time dependence by isolating the time zero mode from the path integral, we contract with the state e^{-qX^0} and then Laplace transform in q to the ‘closed string time’ x^0 , see later. Combining [31] with the appropriate momentum conservation, we get

$$f_\lambda(q) = \left\langle e^{-\lambda \int_0^{2\pi} d\theta e^{X^0}(e^{i\theta})} e^{-qX^0}(0) \right\rangle_{\text{disk}}^{X^0} \quad (3.44)$$

$$= \sum_{n=0}^{\infty} (-\lambda)^n 2\pi\delta(n-q), \quad (3.45)$$

where we took advantage of the disk geometry and the fact that the distance between the boundary insertions and the bulk insertion is always 1.

To find the explicit dependence in time we should in principle Wick rotate, Fourier transform and Wick-rotate back. A tailor-made shortcut for this particular example is just to Laplace transform in real time, without any Wick rotation. In particular we have

$$f_\lambda(x^0) \equiv \int_0^\infty \frac{dk}{2\pi} f_\lambda(q) e^{qX^0} = \frac{1}{1 + \lambda e^{x^0}}. \quad (3.46)$$

We then find

$$\frac{T^{ij}(x^0)}{\text{Vol}_{25}} = -\frac{1}{1 + \lambda e^{x^0}} \delta^{ij} \quad (3.47)$$

$$\frac{T^{00}(x^0)}{\text{Vol}_{25}} = 1. \quad (3.48)$$

This is the usual energy-momentum tensor for a half S-brane exhibiting energy conservation and exponential decay for the pressure.¹¹

¹¹We noticed the following curiosity: if we change the boundary condition on X^0 to Dirichlet $X^0(0, \pi) =$

4 Numerical solutions: Lumps in Siegel gauge

The aim of this section is to show how to construct the boundary state from numerical solutions in the level expansion. Our interest is in the Siegel gauge lump solutions initially studied in [34, 35, 36] and recently constructed to greater accuracy by two of us (M.K, M.S.) [37].

4.1 Moeller–Sen–Zwiebach lump at $R = \sqrt{3}$

In [34], up to $L = 3$, the lump solution along a compact direction X , with radius R , has been given in terms of the towers

$$\begin{aligned} |T_n\rangle &= c_1 \cos\left(\frac{n}{R}X(0)\right) |0\rangle \\ |U_n\rangle &= c_{-1} \cos\left(\frac{n}{R}X(0)\right) |0\rangle \\ |V_n\rangle &= c_1 L_{-2}^{(X)} \cos\left(\frac{n}{R}X(0)\right) |0\rangle \\ |W_n\rangle &= c_1 L'_{-2} \cos\left(\frac{n}{R}X(0)\right) |0\rangle \\ |Z_n\rangle &= c_1 L_{-1}^{(X)} L_{-1}^{(X)} \cos\left(\frac{n}{R}X(0)\right) |0\rangle, \end{aligned} \tag{4.1}$$

in the form

$$|\Psi\rangle = \sum_{n|L \leq 3} (t_n |T_n\rangle + u_n |U_n\rangle + v_n |V_n\rangle + w_n |W_n\rangle + z_n |Z_n\rangle). \tag{4.2}$$

The Virasoro's appearing in the expansion of the solution are purely matter and are split in

$$T_{c=26}^{\text{matter}}(z) = T_{c=1}^{(X)}(z) + T'_{c=25}(z),$$

according to

$$\text{BCFT}_{c=26}^{\text{matter}} = \text{BCFT}_{c=1}^{(X)} \otimes \text{BCFT}'_{c=25}.$$

x^0 then (3.1) is no more a solution, because the boundary operator $e^{X^0} = e^{x^0}$ is now a weight zero *number*. This off-shell string field is however a state in the KBc algebra and we can easily compute its Ellwood invariant with a graviton vertex operator $\mathcal{V}^{ij} = -2c\bar{c}\partial X^i \bar{\partial} X^j$ in quite full generality, [16], to find

$$-2\pi i \langle E[\mathcal{V}^{ij}] | \Psi_\lambda - \Psi_{TV} \rangle^{(X^0 \rightarrow \text{Dir})} = -\frac{1}{1 + w\lambda e^{x^0}} \delta^{ij} \text{Vol}_{25} = T_{\lambda \rightarrow w\lambda}^{ij}(x^0) = T_\lambda^{ij}(x^0 + \log w)$$

where $w \equiv -\frac{d}{dK} F^2(K)|_{K=0} > 0$. Note that although the invariant depends now on the choice of security strip (the ‘solution’ is no more a solution so changing the security strip is no more a gauge transformation), the dependence is physically irrelevant as it can be absorbed in a shift in the marginal parameter and thus a shift in time. This reminds of previous observations on the late time behavior of this solution, [32, 33].

Ghost degrees of freedom are spanned by ghost oscillators. More zero momentum primaries of $\text{BCFT}^{(X)}$ appear at higher levels and a more convenient basis is thus given by oscillators in X -direction, [36, 37]. For the time being we consider (4.2) at radius $R = \sqrt{3}$. The reader can find the coefficients $(t_n, u_n, v_n, w_n, z_n)$ of (4.2) in table 3 of [34]. Our aim is to use the result we derived in section (2.3), which allows to define the boundary state in terms of the primaries that are switched on in the solution. The formula (2.52, 2.58) linearly express the coefficients of the Ishibashi states in terms of the coefficients of the solution.

We will be interested in computing the energy density profile of the lump and its pressure along the direction X on which the lump is forming. These quantities can be easily obtained from generalized Ellwood invariants. As in the rolling tachyon case, instead of tensoring an auxiliary BCFT of $c = 0$, we just change a space direction $Y \equiv X^{25}$ to have Dirichlet boundary condition. Since the numerical solution is not turning on any primary along Y , the coefficients of the Ishibashi states we compute are not affected by this.

For the energy profile we have to compute the following generalized Ellwood invariants

$$E_n \equiv -4\pi i \left\langle E[c\bar{c}\partial X^0 \bar{\partial} X^0 e^{i\frac{nX}{R} + \frac{nY}{R}}] \right| \Psi - \Psi_{TV} \rangle. \quad (4.3)$$

The $n = 0$ contribution is precisely the mass (normalized to 1) of the lower dimensional brane, (it is the coefficient of the Ishibashi state of the zero momentum graviton in the time-time direction). In terms of the momenta E_n , the energy density profile can be defined as a simple Fourier series¹²

$$E(x) \equiv T^{00}(x) = \frac{1}{\pi R} \left(\frac{1}{2} E_0 + \sum_{n=1}^{\infty} E_n \cos \frac{nx}{R} \right). \quad (4.4)$$

If a solution describes a lower dimensional brane sitting at $x = 0$, its energy density profile should be given by

$$E(x) = \delta(x) = \frac{1}{\pi R} \left(\frac{1}{2} + \sum_{n=1}^{\infty} \cos \frac{nx}{R} \right). \quad (4.5)$$

Thus an exact lump solution sitting at $x = 0$ will be characterized by

$$E_n = 1, \quad n = 0, \dots, \infty \quad \text{Exact Lump.} \quad (4.6)$$

¹²This is $T^{00} = \frac{1}{2}(A^{00} - B)$ as defined in (3.19). In the present context, it is easy to check that $A^{00} = -B = E$, see (3.35). In this section we set to unity the volume of CFT' . Moreover we normalize the zero mode in the X CFT with the choice $\langle 0|0 \rangle = R$. With this choice the observables we compute are naturally related to integer numbers.

To compute the pressure we need in addition the coefficient of the Ishibashi state for a zero momentum graviton along the X -direction, which is captured by the Ellwood invariant

$$D \equiv 4\pi i \left\langle E[c\bar{c}\partial X\bar{\partial}X] \middle| \Psi - \Psi_{TV} \right\rangle. \quad (4.7)$$

This quantity measures how much the original Neumann boundary conditions on the X -BCFT are changed to Dirichlet by the solution. If the solution describes Neumann boundary conditions (like the perturbative vacuum) we will have

$$D^N = 1 \sim \left\langle \partial X \bar{\partial} X(0) \right\rangle_{\text{Disk}}^{\text{Neumann}}, \quad (4.8)$$

while for a single lump we should have

$$D^D = -1 \sim \left\langle \partial X \bar{\partial} X(0) \right\rangle_{\text{Disk}}^{\text{Dirichlet}}. \quad (4.9)$$

The pressure transverse to the lump is given by (minus) the half sum of E_0 and D^{13}

$$P \equiv -\frac{1}{2} (D + E_0). \quad (4.10)$$

Thus an exact lump solution will be characterized by

$$P = 0 \quad \text{Exact Lump}, \quad (4.11)$$

the pressure transverse to a D-brane is zero. Let's see how to compute the E_n 's and D . In order to do so it is very convenient to use the conservation laws for the Ellwood invariant. These conservation laws have been first derived in [6] and [26], we re-derived them (together with a few useful ones) in appendix B with a simpler method. In this particular case we can consider

$$\mathcal{V}^h = c\bar{c}V_{\text{CFT}'}^{(1-h)} V_{\text{CFT}_X}^{(h)}$$

to be a total weight zero primary with weight (h, h) in the X direction. Then we have

$$\langle E[\mathcal{V}^h] | U_n \rangle = \langle E[\mathcal{V}^h] | T_n \rangle \quad (4.12)$$

$$\langle E[\mathcal{V}^h] | V_n \rangle = -\left(16h - \frac{1}{2}\right) \langle E[\mathcal{V}^h] | T_n \rangle \quad (4.13)$$

$$\langle E[\mathcal{V}^h] | W_n \rangle = \left(16h - \frac{7}{2}\right) \langle E[\mathcal{V}^h] | T_n \rangle \quad (4.14)$$

$$\langle E[\mathcal{V}^h] | Z_n \rangle = -2\frac{n^2}{R^2} \langle E[\mathcal{V}^h] | T_n \rangle. \quad (4.15)$$

¹³This quantity is $T^{XX} = \frac{1}{2}(A^{XX} + B) = \frac{1}{2}(A^{XX} - A^{00})$, see discussion around (3.19). Its computation proceeds analogously to the T^{00} computed in the previous section for the rolling tachyon, see (3.34). The nonzero momentum part of A^{XX} is computed by writing $\partial X \bar{\partial} X e^{inX/R}$ as a descendent of the tachyon $e^{inX/R}$, which precisely cancel the corresponding contribution from $B = -A^{00}$. So only the zero momentum part is nontrivial and it gives rise to the expression (4.10).

Thus, up to level 3

$$E_n = -4\pi i \left\langle E[c\bar{c}\partial X^0 \bar{\partial} X^0 e^{i\frac{nX}{R} + \frac{nY}{R}}] \middle| \Psi - \Psi_{TV} \right\rangle \quad (4.16)$$

$$= -4\pi i \sum_{m|L \leq 3} f_{nm} \left\langle E[c\bar{c}\partial X^0 \bar{\partial} X^0 e^{i\frac{nX}{R} + \frac{nY}{R}}] \middle| T_m \right\rangle. \quad (4.17)$$

where the coefficients f_{nm} are given by the conservation laws (4.12)

$$f_{nm} = -\delta_{m0} \frac{2}{\pi} + t_m + u_m - \left(4\frac{n^2}{R^2} - \frac{1}{2}\right) v_m + \left(4\frac{n^2}{R^2} - \frac{7}{2}\right) w_m - \frac{2m^2}{R^2} z_m. \quad (4.18)$$

Notice that the tachyon vacuum has been subtracted from the solution by the $-\frac{2}{\pi}|T_0\rangle$ term in f_{nm} . We are left with a single Ellwood invariant whose computation gives, see (2.58),

$$-4\pi i \left\langle E[c\bar{c}\partial X^0 \bar{\partial} X^0 e^{i\frac{nX}{R} + \frac{nY}{R}}] \middle| T_m \right\rangle = -\frac{\pi R}{2} 4^{\frac{n^2}{R^2}} \delta_{nm} \frac{1 + \delta_{n0}}{2}. \quad (4.19)$$

Thus we have

$$E_n = -\frac{\pi R}{2} 4^{\frac{n^2}{R^2}} \frac{1 + \delta_{n0}}{2} f_{nn}. \quad (4.20)$$

The computation of D proceeds analogously but in a simpler way, since only the zero momentum part of the solution participates. Using the conservation laws (4.12) we find

$$\begin{aligned} D &= 4\pi i \left\langle E[c\bar{c}\partial X \bar{\partial} X] \middle| \Psi - \Psi_{TV} \right\rangle \\ &= 4\pi i d_0 \left\langle E[c\bar{c}\partial X \bar{\partial} X] \middle| T_0 \right\rangle \\ &= -\frac{\pi R}{2} d_0, \end{aligned} \quad (4.21)$$

where, up to $L = 3$

$$d_0 = -\frac{2}{\pi} + t_0 + u_0 - \frac{31}{2}v_0 + \frac{25}{2}w_0. \quad (4.22)$$

Using the coefficients given in table 3 of [34] we find the following values

L ($R = \sqrt{3}$)	Action	E_0	E_1	E_2	E_3	D	P
1/3	1.32002	1.23951	0.743681	—	—	1.23951	−1.23951
4/3	1.25373	1.14776	0.741903	0.825738	—	1.14776	−1.14776
2	1.11278	1.10298	0.830459	0.927894	—	−0.574734	−0.264122
7/3	1.07358	1.07489	0.899585	1.0405	—	−0.992768	−0.0410632
3	1.06421	1.0645	0.89973	1.07981	1.23776	−1.08289	0.00919196
Expected	1	1	1	1	1	−1	0

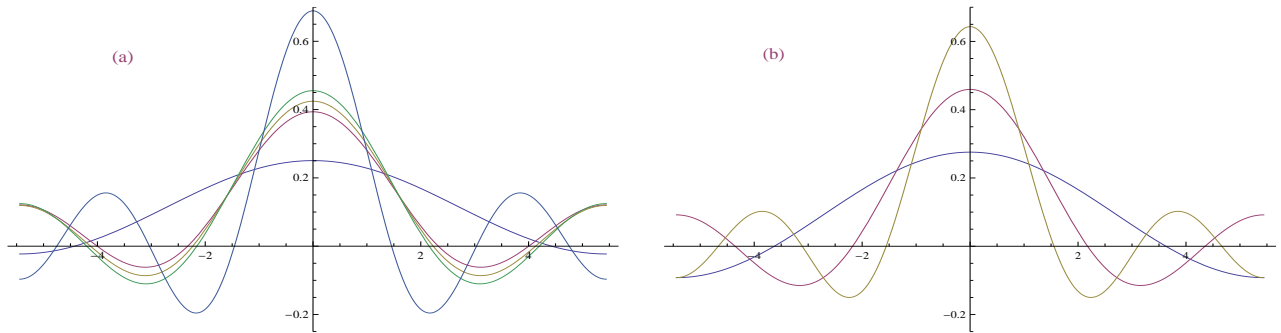


Figure 4.1: (a) Gauge-invariant energy density profile of the Siegel-gauge-lump at $R = \sqrt{3}$, at levels $L = \frac{1}{3}, \frac{4}{3}, 2, \frac{7}{3}, 3$, as defined by eq. (4.4). At level $1/3$ only one harmonic is available and corresponds to the less localized profile. At levels $L = 4/3, 2, 7/3$ the second harmonic enters the game and the profile is essentially unchanged till $L = 3$ where the third harmonic gives a substantial contribution. (b) Plot of $\frac{1}{\pi R} \left(\frac{1}{2} + \sum_{n=1}^N \cos \frac{n}{R} x \right)$, for $N = 1, 2, 3$, at $R = \sqrt{3}$. This is how the delta-function forms in an expansion in harmonics.

In the first column we have also written down the mass of the lump as computed from the action, table 4 ($r^{(1)}$) of [34]. The pressure is nicely going to zero. To give an optical visualization we plot the energy density profile in figure 4.1a. To compare we plot the approximants of the delta function $\frac{1}{\pi R} \left(\frac{1}{2} + \sum_{n=1}^N \cos \frac{n}{R} x \right)$ for $N = 1, 2, 3$, see figure 4.1b. It is also interesting to qualitatively compare with the known open-string-tachyon profile (given by $\sum_n t_n \cos \frac{n}{R} x$, see figure 4.2). It is apparent that in the ‘closed-string’ profile of figure 4.1 the higher harmonics play an essential role in localizing it to zero width, while this does not happen in the open string profile. This is a consequence of the geometry of the identity string field, which effectively dresses the tachyon coefficients t_n with $4 \frac{n^2}{R^2}$ thus amplifying the effect of higher harmonics. As it often happens, subleading contributions in the Fock space can have important sizable effects in observables.

Up to here, we have just used the coefficients given in [34]. There, the maximum level is $L = 3$ and it is thus desirable to have a higher level confirmation that our proposal is indeed computing the correct energy profile, because subtleties related to the identity string field might manifest themselves at higher levels. With the code developed by two of the authors [37] it is possible to go up to $L = 10$ with a reasonable personal computer power and to explore the lump solutions for different radii. Using cluster facilities we arrived to level 12 and, for $R = \sqrt{3}$ in the same $(L, 2L)$ scheme as [34], we found¹⁴

¹⁴At higher levels it is more convenient to span the state space of $\text{BCFT}^{(X)}$ with oscillators acting on momentum modes. The conservation laws that are needed to compute the above invariants are derived in appendix B.

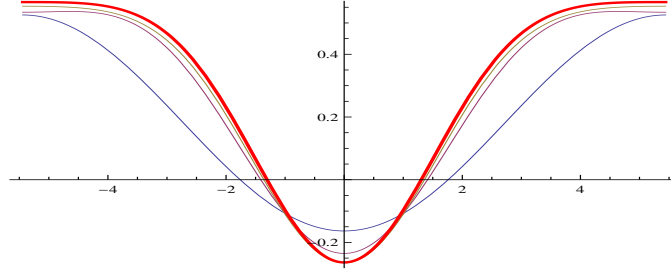


Figure 4.2: Traditional open string tachyon profile of the Siegel-gauge-lump at $R = \sqrt{3}$, at levels $L = \frac{1}{3}$ (blue line), 2 (magenta line), 3 (yellow line) and $L = 12$ (red thick line): higher harmonics are suppressed in the Fock space and the open string profile is essentially unchanged by increasing the level.

L	Action	E_0	E_1	E_2	E_3	E_4	E_5	E_6	D
1	1.32002	1.23951	0.74368	—	—	—	—	—	1.23951
2	1.11278	1.10298	0.83046	0.927897	—	—	—	—	-0.574733
3	1.06421	1.0645	0.899731	1.07981	1.23776	—	—	—	-1.08289
4	1.03731	1.04598	0.917185	0.940436	1.3942	—	—	—	-0.841392
5	1.03006	1.04024	0.939622	0.935288	0.7325	—	—	—	-0.782913
6	1.02141	1.02947	0.945748	0.992221	0.688677	1.9835	—	—	-0.932662
7	1.01893	1.02694	0.956255	0.995945	1.08921	2.07561	—	—	-0.955753
8	1.01477	1.02304	0.959451	0.977503	1.12074	-0.232145	—	—	-0.912786
9	1.01363	1.02183	0.965378	0.977053	0.869997	-0.30867	3.5736	—	-0.91416
10	1.0112	1.01792	0.96745	0.993393	0.86486	1.9131	3.75069	—	-0.961774
11	1.01058	1.01715	0.97137	0.993749	1.04171	1.97574	-3.86516	—	-0.9657950
12	1.008998	1.01550	0.97278	0.98704	1.051325	0.023453	-4.14698	8.073065	-0.94658
Exp.	1	1	1	1	1	1	1	1	-1

The first three lines reproduce the results obtained before using the coefficients of [34]. All the energy harmonics appearing at $L = 3$ show better approximation to the correct value 1. However starting at level 6 the E_4 harmonic enters the game and it will take some more levels for it to start converging to 1. Indeed it appears that higher harmonics oscillate quite erratically before converging to the expected value.¹⁵ We notice that also

¹⁵The convergence of the E_n 's in level truncation does not appear to be uniform and the level at which E_n starts converging increases with n . Technically speaking, the energy profile must be understood as

$$E(x) \equiv \frac{1}{\pi R} \lim_{N \rightarrow \infty} \left[\lim_{L \rightarrow \infty} \left(\frac{1}{2} E_0^{(L)} + \sum_{n=1}^N E_n^{(L)} \cos \frac{nx}{R} \right) \right].$$

In terms of the geometrical definition of the level expansion this means that one would need to consider

$$\lim_{z \rightarrow 1} \lim_{L \rightarrow \infty} \langle E[V] | z^{L_0} | \Psi^{(L)} \rangle,$$

the invariant D (which is a genuine Ellwood invariant) is oscillating with a not so clear pattern with a similar behavior which has been studied in [38].

4.2 Double lumps at $R = 2\sqrt{3}$

For $R > 2$ multiple lump solutions are energetically reachable from the perturbative vacuum via tachyon condensation. As a further application of our formalism we considered double lump solutions. These belong to the family of recently discovered numerical solutions, [37]. The interest here is to show how our gauge invariant expression for the energy density can be used to measure the distance between the lower dimensional branes described by the solution. Another gauge invariant measurement of the distance would be given by the mass of stretched strings between the multiple separated D-branes, which might be harder to measure in the level expansion with enough precision, as it would require a careful study of the linearized fluctuations.

Suppose we have a solution Ψ_a describing two D-branes on a circle of radius R , symmetric around the origin and at a distance a ($2\pi R$) from each other. The energy of the solution will be given by

$$E_0 = 2, \quad (4.23)$$

meaning that we have two lower dimensional branes. But how does the number a show up in the E_n 's? The exact profile of a double lump configuration with separation a ($2\pi R$), centered around πR , is given by

$$E_{(a)}(x) = \delta\left(x - \pi R(1 - a)\right) + \delta\left(x - \pi R(1 + a)\right) = \frac{1}{\pi R} \left(\frac{1}{2} E_0 + \sum_{n=1}^{\infty} E_n \cos \frac{nx}{R} \right). \quad (4.24)$$

Integrating both sides against $\cos \frac{x}{R}$ gives

$$\int_0^{2\pi R} dx \cos \frac{x}{R} E_{(a)}(x) = -2 \cos(\pi a) = E_1. \quad (4.25)$$

Thus, in the case of a two-lump solution, the invariant E_1 measures the distance between the two D-branes

$$a_1 = \frac{1}{\pi} \arccos \left(-\frac{E_1}{2} \right). \quad (4.26)$$

The arc-cosine is defined here in the standard branch $\arccos(0) = \frac{\pi}{2}$. The other branches would give the lengths of all the possible open strings stretching between the branes and

so that we first let the approximate solution $\Psi^{(L)}$ to converge to the exact solution and then we send the regulating strip to zero width. In this way there is a natural cutoff given by $z^{\frac{n^2}{R^2}}$ for higher harmonics.

wrapping the circle at the same time. Higher harmonics can also be used to compute the distance, and integrating (4.24) against $\cos \frac{nx}{R}$ we find

$$E_n = 2(-1)^n \cos(n\pi a) \quad (4.27)$$

Solving this equation for a requires some care in choosing the correct branch of the arc-cosine. This must be done in such a way that the distance computed from any E_n gives the same value a_1 as computed from E_1 . The result can be written as

$$a_n = (-1)^{p_n} \frac{1}{\pi n} \arccos \left((-1)^n \frac{E_n}{2} \right) + \frac{2 \left[\frac{p_n+1}{2} \right]}{n}, \quad n > 1 \quad (4.28)$$

where $[x]$ stands for integer part and the integer p_n is uniquely chosen such that

$$\frac{p_n}{n} < a_1 < \frac{p_n + 1}{n},$$

which gives

$$p_n = [na_1]. \quad (4.29)$$

Clearly for the exact solution Ψ_a we should have

$$a_n = a \equiv \text{Distance}, \quad \forall n \geq 1, \quad (4.30)$$

which is a quite nontrivial constraint between the various E_n , which will be only approximatively satisfied at finite level. For generic multiple lump solutions, the relative distances between the various D-branes can be computed from the E_n -invariants along similar lines. Let's look at a particular example. At level (12, 36) we selected a double lump solution obtained at $R = 2\sqrt{3}$ which displays the open string tachyon profile shown in figure 4.3. The gauge invariant data of the solution are given by¹⁶

¹⁶The solutions at level 2 and 4 (marked in the table with an asterisk) are actually complex. We show the real part of the observables (which would contain a tiny imaginary part of order $10^{-5} - 10^{-1}$ depending on the observable and the level).

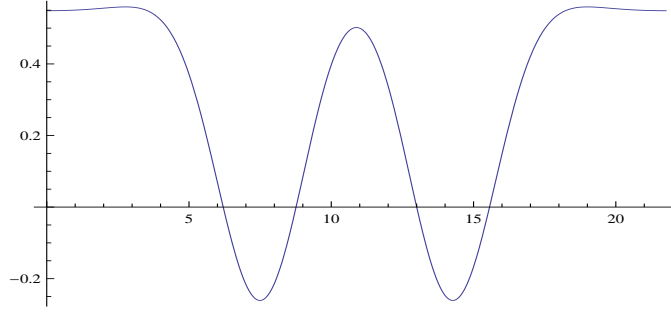


Figure 4.3: Open string tachyon profile of a two-lump solution obtained at $R = 2\sqrt{3}$ and level $L = (12, 36)$.

L	Action	D	E_0	E_1	E_2	E_3	E_4	E_5
1	2.57014	2.4209	2.4209	- 0.816955	- 0.54184	1.3133	-	-
2*	2.21165	-1.69337	2.1897	- 0.848747	- 0.60583	1.89707	- 1.62092	-
3	2.19355	-2.50001	2.11767	- 0.908501	- 0.838798	1.84278	- 1.24372	- 0.987367
4*	2.06874	-1.39183	2.08709	- 0.919667	- 0.850043	1.88425	- 1.0523	- 1.02488
5	2.05531	-1.37542	2.07382	- 0.983959	- 0.812633	1.91245	- 1.15202	- 0.57724
6	2.03894	-2.09185	2.05368	- 1.00138	- 0.788653	1.92175	- 1.30591	- 0.518028
7	2.03494	-2.1419	2.04912	- 1.03283	- 0.765547	1.90846	- 1.35827	- 0.488344
8	2.0269	-1.71527	2.04119	- 1.04599	- 0.743696	1.90879	- 1.35485	- 0.42022
9	2.02525	-1.70495	2.03899	- 1.06273	- 0.734362	1.91644	- 1.37781	- 0.37505
10	2.02052	-2.07063	2.03154	- 1.07229	- 0.717661	1.91526	- 1.44161	- 0.329759
11	2.01969	-2.08504	2.03029	- 1.08369	- 0.709787	1.90937	- 1.45664	- 0.295048
12	2.01658	- 1.81655	2.02687	- 1.09091	- 0.696749	1.90744	- 1.45907	- 0.256288
Expected	2	- 2	2	- 1.18	- 0.61	1.90	- 1.63	0.03

L	E_6	E_7	E_8	E_9	E_{10}	E_{11}	E_{12}
1	-	-	-	-	-	-	-
2*	-	-	-	-	-	-	-
3	2.63667	-	-	-	-	-	-
4*	2.80995	-	-	-	-	-	-
5	1.3382	- 2.40998	-	-	-	-	-
6	1.32239	- 2.61623	- 0.587783	-	-	-	-
7	2.08367	- 0.516486	- 0.299094	4.62486	-	-	-
8	2.07383	- 0.514651	- 0.0446295	4.54688	-	-	-
9	1.60158	- 2.29289	- 0.074498	- 2.51466	- 6.95806	-	-
10	1.58451	- 2.37617	0.281907	- 2.48429	- 7.22557	-	-
11	1.8759	- 1.0672	0.380377	5.38459	8.09111	3.54317	-
12	1.86166	- 1.07601	- 0.0969718	5.28465	8.37884	4.10828	9.35208
Expected	1.60	- 1.92	0.67	1.13	- 2.00	1.23	0.55

The expected values for the E_n 's have been derived using (4.27) from the distance $a_* = 0.299 \pm 0.001$ computed later on. In appendix D another different two-lump solution and a single lump solution are shown for the same value of $R = 2\sqrt{3}$, all results have been pushed to level $L = (12, 36)$. The (E_0, D) invariants and the action are clearly indicating that we are dealing with a two-lump solution.

From E_1 , at the maximal available level $L = 12$, we can compute

$$a_1^{(L=12)} = \frac{1}{\pi} \arccos \left(-\frac{E_1^{(12)}}{2} \right) = 0.316357 \quad (4.31)$$

Looking at figure 4.3 we see that this is consistent with the distance between the minima of the open string tachyon profile, which, at the same level $L = (12, 36)$, is given by

$$a_{open}^{(L=12)} = 0.310439 \quad (4.32)$$

Also E_2 and E_3 gives approximated distances which are quite close to a_{open}

$$a_2^{(L=12)} = \frac{1}{2\pi} \arccos \left(\frac{E_2^{(12)}}{2} \right) = 0.306633 \quad (4.33)$$

$$a_3^{(L=12)} = \frac{1}{3\pi} \arccos \left(-\frac{E_3^{(12)}}{2} \right) = 0.300927. \quad (4.34)$$

Going further with the harmonics we have to change the branch of the arc-cosine, according to (4.28)

$$a_4^{(L=12)} = -\frac{1}{4\pi} \arccos \left(\frac{E_4^{(12)}}{2} \right) + \frac{1}{2} = 0.309934 \quad (4.35)$$

$$a_5^{(L=12)} = -\frac{1}{5\pi} \arccos \left(-\frac{E_5^{(12)}}{2} \right) + \frac{2}{5} = 0.30818 \quad (4.36)$$

$$a_6^{(L=12)} = -\frac{1}{6\pi} \arccos \left(\frac{E_6^{(12)}}{2} \right) + \frac{1}{3} = 0.313486. \quad (4.37)$$

Notice that by (4.28) the E_n 's must be bounded by 2 in absolute value, to be consistent with a two-branes interpretation. The $E_{n \geq 6}$'s already show 'incorrect values' up to level 8 and indeed they would need higher level to start showing a convergence pattern. The set of candidate gauge invariant distances $(a_1, a_2, a_3, a_4, a_5)$ is plotted in figure 4.4 for the range of levels $L = 2, 3, \dots, 12$. To obtain a prediction on the actual distance between the two D-branes, we have to find a way to extrapolate the observables of the approximate

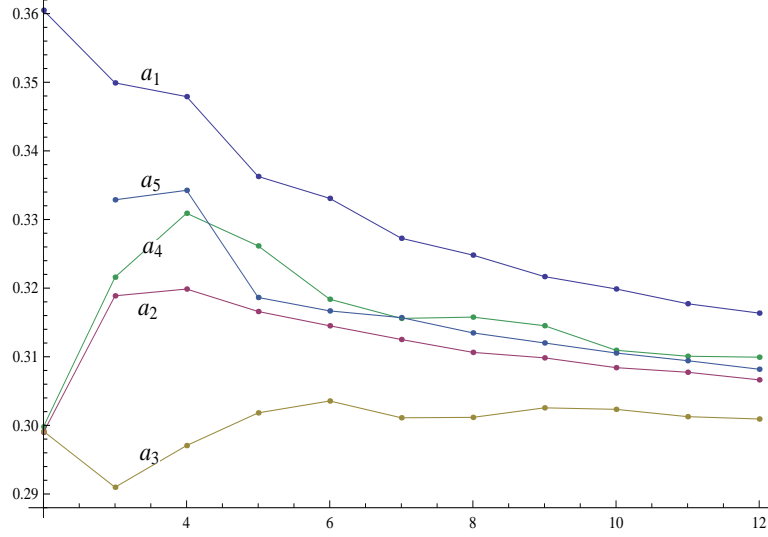


Figure 4.4: Gauge invariant distances $a_n^{(L)}$ as computed from the first five E_n -invariants, as a function of the level ($L = 2, 3, \dots, 12$). Notice that starting from $L = 4$ the a_n 's show a quite clear (but not so fast) convergence pattern.

level-truncated solution to infinite level. Rather surprisingly, we observed that a simple $1/L$ fit gives results which are nicely consistent within the first five harmonics. For any harmonic $n = 1, \dots, 5$ we fitted the values of $a_n^{(L)}$

$$a_n^{(L)} \approx a_n^{(\infty, L_{min})} + \frac{b_n^{(\infty, L_{min})}}{L}, \quad (4.38)$$

in the range of levels ($L_{min}, L_{max} = 12$) for all possible L_{min} 's in the range

$$4 \leq L_{min} \leq (L_{max} - 2) = 10.$$

The lower bound $4 \leq L_{min}$ is justified by figure 4.4 and the upper bound is necessary to have at least 3 points to fit. See figure 4.5 for an example.

By varying L_{min} one can have an estimate of the error of the linear fit. For any frequency n we take the mean value $\bar{a}_n^{(\infty)}$ from the results obtained for different L_{min} and we take the associated standard deviation σ_n as a measure of the error. The obtained values are the following

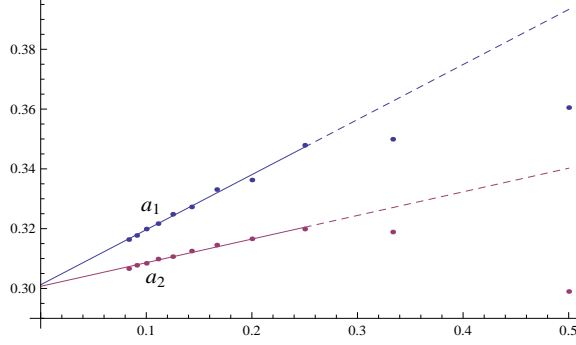


Figure 4.5: Plot of the distances a_1 and a_2 as functions of $1/L$, together with their best fit $a_n^{(\infty, L_{min})} + \frac{b_n^{(\infty, L_{min})}}{L}$. The fit here is done in the range $L = 4, \dots, 12$, i.e. $L_{min} = 4$, and it appears to essentially capture the dependence of the a_n 's with the level.

n	$\bar{a}_n^{(\infty)}$	σ_n
1	0.300	0.001
2	0.299	0.001
3	0.299	0.004
4	0.299	0.003
5	0.298	0.002

These are five independent ‘measurements’ of the distance and the fact that they are all mutually consistent is quite a nontrivial check for the linear fit. Given the mutual consistency between these values, we can average them with weights $w_n = 1/\sigma_n^2$ and obtain the value for the distance

$$a_* = 0.299 \pm 0.001, \quad (4.39)$$

where the error has been computed with $(\sum_n w_n)^{-1/2}$.

Since we are ‘measuring’ a modulus of a BCFT in an unknown point of its moduli space via an approximated OSFT solution, we do not have a given value to compare with, but to appreciate to what extent $a_* \sim 0.3$ is consistent with the distance between the two D-branes described by the solution, we plot the energy profile of the solution including up to 6th harmonic against the corresponding truncation of a sum of two delta functions, at distance $a = 0.3$, see figure 4.6.

For completeness, few more calculations are needed to completely reconstruct the boundary state for a system of parallel lower dimensional branes. At zero momentum we have an infinite tower of bulk primaries with weight (h^2, h^2) for integer h . Here we have only considered the coefficients of the Ishibashi states for the identity (captured by

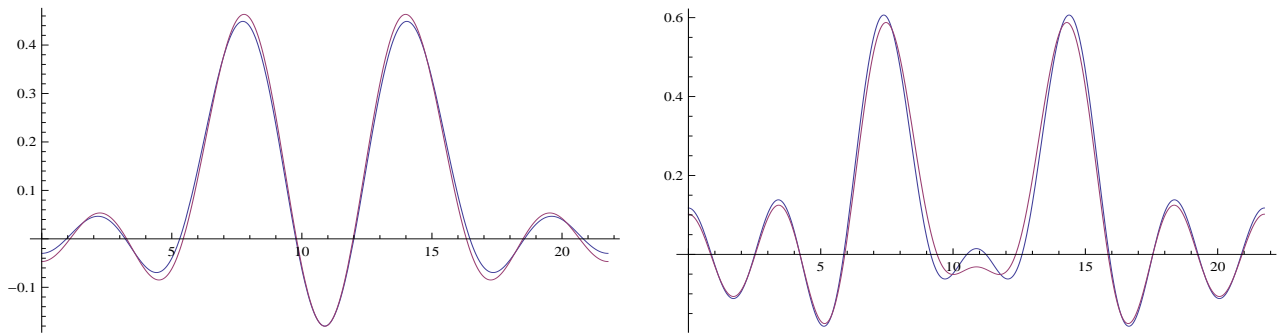


Figure 4.6: Plot of $\frac{1}{\pi R} \left(\frac{1}{2} E_0 + \sum_{n=1}^m E_n \cos \frac{nx}{R} \right)$, in blue line, against the corresponding truncation of a sum of two delta functions, in magenta line, at a separation $\bar{a}_* = 0.3$, for $m = 4$ (left) and $m = 6$ (right). Here $R = 2\sqrt{3}$ and the coefficients are obtained at level $L = (12, 36)$. Notice how the profile of the truncated solution displays a slightly bigger effective distance than the ‘exact’ one, as if the lumps were getting closer by increasing the level.

E_0) and for the weight (1,1) primary $\partial X \bar{\partial} X$ (captured by D). We didn’t compute the coefficients of the Ishibashi states of the higher level zero momentum primaries. It would also be necessary to verify that the coefficients of the winding modes Ishibashi states are vanishing, as it must be for Dirichlet boundary conditions. As for the zero momentum primaries, these coefficients too get contribution only from the zero momentum part of the solution. We leave these computations for future work.

5 Conclusions

In this paper we have proposed and analyzed a simple shortcut to compute the BCFT boundary state corresponding to a classical solution of OSFT. In a nutshell, OSFT provides the coefficients for the linear combination of Ishibashi states forming the boundary state. These coefficients are given by Ellwood invariants of a modified OSFT where a trivial $\text{BCFT}^{(\text{aux})}$ with $c = 0$ has been tensored with the original BCFT_0 . In most cases it is however sufficient to assign one space-time direction Dirichlet boundary conditions. The essence of the trick is to associate to any closed string primary of the form $c\bar{c}V^{\text{matter}}$ a corresponding weight zero primary with nonvanishing tadpole thanks to the Dirichlet boundary condition, without altering the physics described by the solution (the solution remains a solution in the tensor theory). Assuming Ellwood conjecture thus completely defines the boundary state.

Given any solution Ψ , a family of closed string states $|B_*(\Psi)\rangle^{(s)}$ has been constructed in [4]. These states are conjectured to be BRST-equivalent to the boundary state we deal

with in this paper. Among other things, the construction depends on a free parameter s and, in the $s \rightarrow 0$ limit, under some assumptions on the regularity of the solution, one recovers the worldsheet geometry of the Ellwood invariant. Our construction, where spinless matter primaries are lifted to formal elements of the closed string cohomology in an enlarged CFT, applies to [4] as well. All the derivations of [4] go through in the tensor theory and, inside the (enlarged) closed string cohomology, they clearly agree with our general results of section 2, assuming Ellwood conjecture. Thus, in principle, the coefficients of the Ishibashi states (2.34), and hence the full BCFT-boundary state, can be computed as well from

$$n_{\Psi}^{\alpha} = -\frac{1}{2} \langle \tilde{\mathcal{V}}^{\alpha} | (c_0 - \bar{c}_0) | B_*(\tilde{\Psi}) \rangle^{(s)}. \quad (5.1)$$

The construction at finite s is a gauge invariant deformation/regularization of the generalized Ellwood invariant. The final s -independence of this deformation is a consequence of closed string linearized gauge invariance (the dependence of $|B_*(\tilde{\Psi})\rangle^{(s)}$ on s is, in general, at most BRST exact) and, ultimately, of the validity of the OSFT equation of motion, [4]. This is an important point given that there are in general infinite families of string fields sharing the same Ellwood invariants with OSFT solutions.

Defining the boundary state from OSFT can potentially contribute to the development of boundary conformal field theory. The classification of consistent boundary states in a given CFT background is, at present, still an open problem. Our proposal gives a complementary way of determining the coefficients of the Ishibashi states, without having to deal with complicated consistency conditions such as Cardy or various sewing conditions [12]. What we have to do, instead, is to *solve* the OSFT equation of motion. This, at least in principle, is a clear well defined task and we have at our disposal a lot of analytic and numerical tools for progressing in this direction.

From the OSFT point of view, it is important to understand if the space of classical solutions is bigger or smaller than the space of consistent boundary conditions. Suppose OSFT has more solutions than expected: this, for example, could show up in exotic “non integer” values for the Ellwood invariants (an example of this possibility has been found [39], in the context of cubic super string field theory) and hence our boundary state would violate Cardy conditions. On the other hand, our method could prove useful in the search for solutions associated with generic BCFT’s whose boundary state is given. Matching the generalized Ellwood invariants of a to-be-found solution to a target boundary state with given BCFT moduli, will partially constrain the coefficients of the solution and a full solution can in principle be searched for in the level expansion (or in some other regularization) by extremizing the constrained action. Consequently one must verify that the full action is also extremized, along the lines of [40]. However it is not guaranteed,

see for example [41], that such a solution would exist for any choice of the BCFT moduli.

Acknowledgments

We would like to thank Ted Erler, Mathias Gaberdiel and Yuji Okawa for useful discussions. The access to computing and storage facilities owned by parties and projects contributing to the Czech National Grid Infrastructure MetaCentrum, provided under the programme "Projects of Large Infrastructure for Research, Development, and Innovations" (LM2010005), and to CERIT-SC computing facilities provided under the programme Center CERIT Scientific Cloud, part of the Operational Program Research and Development for Innovations, reg. no. CZ. 1.05/3.2.00/08.0144 is highly appreciated. This research was supported by the EURYI grant GACR EYI/07/E010 from EURO-HORC and ESF.

A BCFT^(aux)

An explicit example of BCFT^(aux) is given by tensoring a free boson with Dirichlet boundary condition ($c = 1$) with an appropriate $c = -1$ BCFT.

For the first Dirichlet factor it is useful to recall the following one point function (in units $\alpha' = 1$)

$$\langle : e^{qX} : (z, \bar{z}) \rangle_{\text{disk}}^{\text{Dirichlet}} = (1 - |z|^2)^{\frac{q^2}{2}} \quad (\text{A.1})$$

Here we take q to be a generic complex number. For q imaginary, the vertex operator has positive weight. For q real it is a negative weight primary and it is not normalizable as a free field. However in the presence of Dirichlet boundary conditions, the divergence of the zero mode integration is weaker than the delta function imposed by the boundary condition. To derive (A.1) we start with the Green's function on the disk with Dirichlet boundary conditions.

$$G(z, w) = \langle X(z, \bar{z}) X(w, \bar{w}) \rangle_{\text{disk}}^{\text{Dirichlet}} = -\frac{1}{2} \log |z - w|^2 + \frac{1}{2} \log |1 - \bar{z}w|^2, \quad (\text{A.2})$$

where the disk is defined by $|z| \leq 1$. One can easily check that

$$G(e^{i\theta}, w) = G(z, e^{i\theta}) = 0, \quad (\text{A.3})$$

meaning that the Dirichlet condition $X(|z| = 1) = 0$ is satisfied. The first term is just the usual Green's function on the complex plane (no boundary) while the second is the effect of the Dirichlet boundary conditions. Only the first (bulk) term in (A.2) should be

subtracted for closed string vertex operators. Then the result (A.1) easily follows from usual Wick contractions.

An example of the needed $c = -1$ theory can be taken to be a free boson ϕ with background charge Q with an energy momentum tensor given by

$$T(z) = - : \partial\phi\partial\phi : + iQ\partial^2\phi \quad (\text{A.4})$$

The central charge is given by

$$c = 1 - 6Q^2. \quad (\text{A.5})$$

So we have $c = -1$ for $Q = \frac{1}{\sqrt{3}}$. The weight of a primary field is

$$h[e^{i\alpha\phi}] = \frac{\alpha}{2} \left(\frac{\alpha}{2} - q \right) = \frac{\alpha}{2} \left(\frac{\alpha}{2} - \frac{1}{\sqrt{3}} \right). \quad (\text{A.6})$$

Notice that, differently from the $Q = 0$ case there are now two operators with vanishing weight, one being the identity operator and the other being

$$w(z) = e^{\frac{2i}{\sqrt{3}}\phi(z)}, \quad (\text{A.7})$$

which is in general needed to screen the background charge. The boundary conditions in the upper half plane are of Neumann type

$$\phi(z) = \bar{\phi}(\bar{z}), \quad z = \bar{z} \quad (\text{A.8})$$

We normalize w such that the disk correlator is given by

$$\langle w(z) \rangle_{\text{disk}} = \langle \bar{w}(\bar{z}) \rangle_{\text{disk}} = \langle w(\xi) \rangle_{\text{UHP}} = \langle \bar{w}(\bar{\xi}) \rangle_{\text{UHP}} = 1, \quad \forall z, \xi. \quad (\text{A.9})$$

Another simple option for BCFT^{aux} is to choose a product of two free bosons ($c = 2$) with Dirichlet boundary conditions, and the symplectic fermion theory with $c = -2$ constructed in [42].¹⁷ The advantage of such a choice is that we do not need the analogue of w to saturate the zero modes on the disk.

B Conservation laws for the Ellwood invariant

In this appendix we derive a set of useful conservation laws for the Ellwood invariant. Some of them have been derived already in [6] and [26]. We present an alternative simpler derivation.

¹⁷Further details and references are summarized in [43].

B.1 Review of conservation laws of the identity string field

To start with, it is useful to recall some standard conservation laws for the identity string field $|I\rangle$ (see [9, 44] for review). They will be later modified by the closed string states at the midpoint in the Ellwood invariant. The first one is the anomalous conservation of K_n , for nonvanishing central charge c ,

$$K_n = L_n - (-1)^n L_{-n}, \quad (\text{B.1})$$

$$\langle I|K_n = \frac{c}{8}n(i^n + (-i)^n)\langle I|. \quad (\text{B.2})$$

Another useful conservation law is given by the modes of current $i\partial X$, which reads

$$A_n = \alpha_n + (-1)^n \alpha_{-n}, \quad (\text{B.3})$$

$$\langle I|A_n = 0. \quad (\text{B.4})$$

Because of non-anomalous momentum conservation in the X -CFT, this conservation law shows no anomaly. Let's come to the ghost sector. From the conservation of K_n we can read-off, by analogy, the conservation of B_n , which is not anomalous ($b(z)$ is a genuine weight two primary)

$$B_n = b_n - (-1)^n b_{-n}, \quad (\text{B.5})$$

$$\langle I|B_n = 0. \quad (\text{B.6})$$

There is also an analogous (anomalous) conservation for the c -ghost which reads

$$C_n = c_n + (-1)^n c_{-n}, \quad (\text{B.7})$$

$$\langle I|C_{2n} = -(-1)^n \langle I|C_0, \quad (\text{B.8})$$

$$\langle I|C_{2n+1} = -(-1)^n \langle I|C_1. \quad (\text{B.9})$$

B.2 Virasoro conservation laws

In this section we compute general conservation laws for Virasoro generators. Suppose that the total CFT is the tensor product $\text{CFT}^{(1)} \otimes \text{CFT}^{(2)}$, of two CFT's with central charges c and $-c$ respectively. The energy-momentum tensor decomposes as

$$T(z) = T^{(1)}(z) + T^{(2)}(z). \quad (\text{B.10})$$

The weight-zero vertex operator entering the Ellwood invariant can be written as

$$V(z, \bar{z}) = V_{(1)}^{(h, \bar{h})} V_{(2)}^{(-h, -\bar{h})}(z, \bar{z}), \quad (\text{B.11})$$

where h and \bar{h} are the holomorphic and antiholomorphic weights (not necessarily the same) of $V_{(1)}$ with respect to $T^{(1)}$. The corresponding BRST and conformal invariant Ellwood state is given by

$$\langle E[V] | = \langle I | V_{(1)}^{(h, \bar{h})} V_{(2)}^{(-h, -\bar{h})}(i, -i). \quad (\text{B.12})$$

We start the computation of the conservation law for the modes of $T^{(1)}$, using the anomalous derivation

$$K_n^{(1)} = L_n^{(1)} - (-1)^n L_{-n}^{(1)} = \oint \frac{dw}{2\pi i} v_n(w) T^{(1)}(w),$$

where the holomorphic vector field $v_n(w)$ is given by

$$v_n(w) = w^{n+1} - (-1)^n w^{-n+1}. \quad (\text{B.13})$$

We have

$$\begin{aligned} \langle E[V] | K_n^{(1)} &= \langle I | V_{(1)}^{(h, \bar{h})} V_{(2)}^{(-h, -\bar{h})}(i, -i) K_n^{(1)} \\ &= \oint_0 \frac{dw}{2\pi i} v_n(w) \langle I | V_{(1)}^{(h)}(i) \bar{V}_{(1)}^{(\bar{h})}(-i) V_{(2)}^{(-h, -\bar{h})}(i, -i) T^{(1)}(w). \end{aligned} \quad (\text{B.14})$$

Now, using the formalism of [45, 9], we write $\langle I | = \langle 0 | U_f$, with

$$f(w) = \frac{2w}{1 - w^2}$$

being the identity conformal map and we move the operator U_f to the right of the other operators.

$$\begin{aligned} &\oint_0 \frac{dw}{2\pi i} v_n(w) \langle I | V_{(1)}^{(h)}(i) \bar{V}_{(1)}^{(\bar{h})}(-i) V_{(2)}^{(-h, -\bar{h})}(i, -i) T^{(1)}(w) \\ &= \oint_0 \frac{dw}{2\pi i} v_n(w) \langle 0 | V_{(1)}^{(h)}(i) \bar{V}_{(1)}^{(\bar{h})}(-i) \left([f'(w)]^2 T^{(1)}(f(w)) + \frac{c}{12} S_f(w) \right) V_{(2)}^{(-h, -\bar{h})}(i, -i) U_f. \end{aligned}$$

At this point we notice that the geometry of the identity string field, together with the involved operator insertions at the midpoint, implies that

$$\oint_0 = -\frac{1}{2} \oint_{(i, -i)}. \quad (\text{B.15})$$

To see this we start with the simple observation that the integrand has only poles in $(0, \pm i, \infty)$. The poles at the midpoint arise from the $T - V$ contractions and from the Schwarzian derivative

$$S_f(w) = \{f(w), w\} = \frac{6}{(1 + w^2)^2}. \quad (\text{B.16})$$

Then we notice

$$f\left(-\frac{1}{w}\right) = f(w) \quad (\text{B.17})$$

$$f'\left(-\frac{1}{w}\right) = w^2 f'(w) \quad (\text{B.18})$$

$$v_n\left(-\frac{1}{w}\right) = \frac{1}{w^2} v_n(w) \quad (\text{B.19})$$

$$S_f\left(-\frac{1}{w}\right) = w^4 S_f(w), \quad (\text{B.20})$$

from which it follows from contour deformation that

$$\oint_0 = \oint_\infty = -\frac{1}{2} \oint_{(i,-i)}.$$

We thus get

$$\begin{aligned} \langle E[V] | K_n^{(1)} \rangle &= -\frac{1}{2} \oint_{(i,-i)} \frac{dw}{2\pi i} v_n(w) [f'(w)]^2 \langle 0 | V_{(1)}^{(h)}(i) \bar{V}_{(1)}^{(\bar{h})}(-i) T^{(1)}(f(w)) V_{(2)}^{(-h,-\bar{h})} U_f \\ &\quad - \frac{c}{24} \oint_{(i,-i)} \frac{dw}{2\pi i} v_n(w) S_f(w) \langle 0 | V_{(1)}^{(h)}(i) \bar{V}_{(1)}^{(\bar{h})}(-i) V_{(2)}^{(-h,-\bar{h})}(i, -i) U_f \\ &= \frac{1}{2} \oint_i \frac{dw}{2\pi i} v_n(w) [f'(w)]^2 \langle 0 | \left[\frac{h V_{(1)}^{(h)}(i)}{(f(w) - i)^2} + \frac{\partial V_{(1)}^{(h)}(i)}{f(w) - i} \right] \bar{V}_{(1)}^{(\bar{h})}(-i) V_{(2)}^{(-h,-\bar{h})}(i, -i) U_f \\ &\quad + \frac{1}{2} \oint_{-i} \frac{dw}{2\pi i} v_n(w) [f'(w)]^2 \langle 0 | V_{(1)}^{(h)}(i) \left[\frac{\bar{h} \bar{V}_{(1)}^{(\bar{h})}(-i)}{(f(w) + i)^2} + \frac{\partial V_{(1)}^{(h)}(-i)}{f(w) + i} \right] V_{(2)}^{(-h,-\bar{h})}(i, -i) U_f \\ &\quad - \frac{c}{24} \oint_{(i,-i)} \frac{dw}{2\pi i} v_n(w) S_f(w) \langle 0 | V_{(1)}^{(h)}(i) \bar{V}_{(1)}^{(\bar{h})}(-i) V_{(2)}^{(-h,-\bar{h})}(i, -i) U_f. \end{aligned}$$

It is easy to see that the terms proportional to the non primary operators ∂V gives vanishing contribution.¹⁸ Everything thus simplifies down to

$$\begin{aligned} &= -\frac{1}{2} \oint_{(i,-i)} \frac{dw}{2\pi i} v_n(w) \left[\frac{c}{12} S_f(w) + [f'(w)]^2 \left(\frac{h}{(f(w) - i)^2} + \frac{\bar{h}}{(f(w) + i)^2} \right) \right] \langle E[V] | \\ &= n \left[i^n \left(\frac{c}{8} - 4h \right) + (-i)^n \left(\frac{c}{8} - 4\bar{h} \right) \right] \langle E[V] |. \end{aligned} \quad (\text{B.21})$$

¹⁸This would not be the case if we worked in the geometry of the local coordinate w where we would have ended with the singular insertion $\sim v_n(i) \langle I | \partial V(i) \sim 0 \times \infty$. In this case one would need to displace the insertion a bit away from the midpoint and send the regulator to zero after taking the residue. This gives a finite net contribution which adds up to the naive contribution from the double pole. In the global coordinate geometry $\tilde{w} = f(w)$ no regularization is needed and the total contribution just come from the “double pole” $\sim \frac{1}{(f(w) \pm i)^2}$. Similar considerations apply to the other conservation laws we discuss next.

Summarizing, we found

$$\boxed{\langle E[V_{(1)}V_{(2)}] | K_n^{(1)} = n \left[i^n \left(\frac{c}{8} - 4h \right) + (-i)^n \left(\frac{c}{8} - 4\bar{h} \right) \right] \langle E[V_{(1)}V_{(2)}] |} \quad (\text{B.22})$$

The conservation law for $K_n^{(2)}$ is simply obtained by changing $c \rightarrow -c$ and $(h, \bar{h}) \rightarrow (-h, -\bar{h})$.

B.3 Oscillator conservation laws

It is useful to derive the conservation laws for the current $i\sqrt{2}\partial X$ of a free boson. We focus on Ellwood states with two kinds of closed string vertex operators: pure momentum modes and the zero momentum primary $\partial X \bar{\partial} X$. First we compute the conservation laws of α oscillators acting on momentum modes. We define

$$A_n = \alpha_n + (-1)^n \alpha_{-n} = \oint \frac{dw}{2\pi i} g_n(w) i\sqrt{2}\partial X(z). \quad (\text{B.23})$$

The function $g_n(w)$ is defined as

$$g_n(w) = w^n + (-1)^n w^{-n}, \quad (\text{B.24})$$

and obeys

$$g_n\left(-\frac{1}{w}\right) = g_n(w). \quad (\text{B.25})$$

Acting with A_n on a Ellwood state of definite momentum¹⁹

$$\begin{aligned} \langle I | e^{ikX}(i, -i) V_{(2)}(i, -i) A_n &= i\sqrt{2} \oint_0 \frac{dw}{2\pi i} g_n(w) \langle I | e^{ikX}(i, -i) V_{(2)}(i, -i) \partial X(w) \\ &= i\sqrt{2} \oint_0 \frac{dw}{2\pi i} g_n(w) f'(w) \langle 0 | e^{ikX}(i, -i) \partial X(f(w)) V_{(2)}(i, -i) U_f. \end{aligned} \quad (\text{B.26})$$

As in the previous section, here again we notice that, because of (B.25) we can substitute

$$\oint_0 \rightarrow -\frac{1}{2} \oint_{(i, -i)}.$$

Using the OPE

$$e^{ikX}(i, -i) \partial X(f(w)) \sim -\frac{ik}{2} \left(\frac{1}{f(w) - i} + \frac{1}{f(w) + i} \right) e^{ikX}(i, -i), \quad (\text{B.27})$$

¹⁹The plane wave e^{ikX} is supplemented with a primary in a decoupled sector of weight $-\frac{k^2}{4}$, which we call $V_{(2)}$.

and taking the residues at the midpoints we are left with the simple result

$$\langle I | e^{ikX} V_{(2)}(i, -i) A_n = -(i^n + (-i)^n) \sqrt{2} k \langle I | e^{ikX} V_{(2)}(i, -i) \rangle \quad (\text{B.28})$$

In addition, we need the conservation laws for the Ellwood invariant given by the zero momentum graviton vertex operator $c\bar{c}\partial X\bar{\partial}X$.

$$\langle I | c\bar{c}(i, -i) \partial X \bar{\partial} X(i, -i) A_n \quad (\text{B.29})$$

$$= i\sqrt{2} \oint_0 \frac{dw}{2\pi i} g_n(w) \langle I | \partial X(i) \partial X(-i) \partial X(w) c\bar{c}(i, -i) \rangle \quad (\text{B.30})$$

$$\begin{aligned} &= i\sqrt{2} \oint_0 \frac{dw}{2\pi i} g_n(w) f'(w) \langle 0 | \partial X(i) \partial X(-i) \partial X(f(w)) c\bar{c}(i, -i) U_f \\ &= i\sqrt{2} \left(-\frac{1}{2}\right) \oint_{(i, -i)} \frac{dw}{2\pi i} g_n(w) f'(w) \langle 0 | \partial X(i) \partial X(-i) \partial X(f(w)) c\bar{c}(i, -i) U_f \\ &= -\frac{i}{\sqrt{2}} \oint_{(i, -i)} \frac{dw}{2\pi i} g_n(w) f'(w) \langle 0 | \left(-\frac{1}{2} \frac{\partial X(-i)}{(f(w) - i)^2} - \frac{1}{2} \frac{\partial X(i)}{(f(w) + i)^2} \right) c\bar{c}(i, -i) U_f. \end{aligned}$$

Taking the residues at the midpoints we get

$$\langle I | c\bar{c}\partial X\bar{\partial}X(i, -i) A_n = \sqrt{2} n^2 (-i)^n \langle 0 | c\bar{c}(i, -i) \left(\partial X(i) - (-1)^n \partial X(-i) \right) U_f \rangle \quad (\text{B.31})$$

Notice that we cannot take the U_f operator back the vacuum because the leftover insertion at the midpoint has overall negative weight. Applying another A_m we get rid of the ∂X insertion and we get

$$\langle I | c\bar{c}\partial X\bar{\partial}X(i, -i) A_n A_m = -2(nm)^2 i^{n+m} ((-1)^n + (-1)^m) \langle 0 | c\bar{c}(i, -i) U_f \rangle. \quad (\text{B.32})$$

Further applications of A_n give trivially zero.

B.4 Ghost conservation laws

The conservation law for b_n oscillators is easily obtained from

$$\begin{aligned} \langle I | c(i) c(-i) V^{(1,1)}(i, -i) B_n &= \oint_0 \frac{dw}{2\pi i} v_n(w) \langle I | c(i) c(-i) V^{(1,1)}(i, -i) b(w) \\ &= \oint_0 \frac{dw}{2\pi i} v_n(w) [f'(w)]^2 \langle 0 | c(i) c(-i) b(f(w)) V^{(1,1)}(i, -i) U_f \\ &= -\frac{1}{2} \oint_{(i, -i)} \frac{dw}{2\pi i} v_n(w) [f'(w)]^2 \langle 0 | c(i) c(-i) b(f(w)) V^{(1,1)}(i, -i) U_f, \end{aligned} \quad (\text{B.33})$$

where we used

$$\oint_0 = -\frac{1}{2} \oint_{(i,-i)},$$

which is a general property of the identity conservation laws we consider. Performing the midpoint contractions between $b(f(w))$ and $c(\pm i)$, no residue is found ($v_n(\pm i) = 0$) and we are left with the simple

$$\boxed{\langle I|c(i)c(-i)V^{(1,1)}(i,-i)B_n = 0} \quad (\text{B.34})$$

The c ghost conservation law is just a bit more complicated. What happens here is that the anomalies in the conservation of C_n on the identity, (B.8) and (B.9), are killed by the $c\bar{c}(i,-i)$ from the closed string insertion, as we are now going to see.

$$\begin{aligned} \langle I|c(i)c(-i)V^{(1,1)}(i,-i)C_n &= \oint \frac{dw}{2\pi i} h_n(w) \langle I|c(i)c(-i)V^{(1,1)}(i,-i)c(w) \\ &= \oint_0 \frac{dw}{2\pi i} h_n(w) [f'(w)]^{-1} \langle 0|c(i)c(-i)c(f(w))V^{(1,1)}(i,-i)U_f \\ &= -\frac{1}{2} \oint_{(i,-i)} \frac{dw}{2\pi i} h_n(w) [f'(w)]^{-1} \langle 0|c(i)c(-i)c(f(w))V^{(1,1)}(i,-i)U_f. \end{aligned} \quad (\text{B.35})$$

Here the quadratic differential $h_n(w)$ is given by

$$h_n(w) = w^2 (w^n + (-1)^n w^{-n}), \quad (\text{B.36})$$

and obeys

$$h_n\left(-\frac{1}{w}\right) = w^4 h_n(w). \quad (\text{B.37})$$

Once more, this property allows to replace

$$\oint_0 \rightarrow -\frac{1}{2} \oint_{(i,-i)}$$

in going from the second to third line of (B.35). Computing the residues at the midpoint we are left with

$$\begin{aligned} &\langle I|c(i)c(-i)V^{(1,1)}(i,-i)C_n \\ &= -i^{n+1} \langle 0|c(i)c(-i) \left(c(i) - (-1)^n c(-i) \right) V^{(1,1)}(i,-i)U_f. \end{aligned} \quad (\text{B.38})$$

Notice that C_n has been localized to a midpoint insertion in the global coordinate. It is then killed by the two c 's in the closed string state. Thus, differently from the naked identity string field, the conservation law of the c ghost on the Ellwood state is not anomalous.

C General properties of the boundary state

In string theory, the boundary state is a ghost number 3 closed string state and it appears as a source term in the closed string field theory action via the coupling

$$\langle B | c_0^- | \Phi \rangle,$$

where Φ is a dynamical closed string field of total ghost number two. In order to write down a kinetic term for the closed string field, it is necessary to assume the level matching conditions, [46]

$$L_0^- | \Phi \rangle = b_0^- | \Phi \rangle = 0. \quad (\text{C.1})$$

It does not appear consistent to include non level matched closed string states and thus we must also have

$$L_0^- | B \rangle = b_0^- | B \rangle = 0. \quad (\text{C.2})$$

The boundary state is not just a source term in the closed string action but it is also a peculiar state which incarnates the existence of a boundary in CFT_0 (on which we define closed string field theory), which preserves conformal invariance. Together with the previous conditions this means

$$b_0^- | B \rangle = 0 \quad (\text{C.3})$$

$$(L_n^{\text{tot}} - \bar{L}_{-n}^{\text{tot}}) | B \rangle = 0 \quad (\text{C.4})$$

$$(Q_{gh} - 3) | B \rangle = 0, \quad (\text{C.5})$$

where

$$Q_{gh} = \oint \frac{dz}{2\pi i} (- : bc :)(z) + h.c. \quad (\text{C.6})$$

is the total ghost number.

C.1 Proof of matter ghost factorization

Commuting (C.3) with (C.4) we learn that

$$(b_n - \bar{b}_{-n}) | B \rangle = 0, \quad \forall n. \quad (\text{C.7})$$

The most general state obeying (C.7) can be written in normal ordered form as

$$| B \rangle = f\left(\{b_{-m}\}, \{\bar{b}_{-m}\}, c_0^+, [\text{matter}]\right) \exp\left(-\sum_{n=1}^{\infty} \bar{b}_{-n} c_{-n} + b_{-n} \bar{c}_{-n}\right) c_1 \bar{c}_1 | 0 \rangle_{SL(2,C)}, \quad (\text{C.8})$$

where f is a generic function depending on b creation operators, c_0^+ and generic matter operators. To see that this is the case focus on the dependence on $(b_{-n}, \bar{b}_{-n}, c_{-n}, \bar{c}_{-n})$ for fixed $n \geq 1$. Then conditions (C.7) are equivalent to the differential equations

$$(\partial_{c_{-n}} - \bar{b}_{-n})B(c_{-n}, \bar{c}_{-n}, b_{-n}, \bar{b}_{-n}) = 0 \quad (\text{C.9})$$

$$(\partial_{\bar{c}_{-n}} - b_{-n})B(c_{-n}, \bar{c}_{-n}, b_{-n}, \bar{b}_{-n}) = 0, \quad (\text{C.10})$$

whose generic solution is

$$B(c_{-n}, \bar{c}_{-n}, b_{-n}, \bar{b}_{-n}) = f(b_{-n}, \bar{b}_{-n}) \exp(-\bar{b}_{-n}c_{-n} - b_{-n}\bar{c}_{-n}). \quad (\text{C.11})$$

Repeating this procedure for every $n \geq 1$ and also for $n = 0$, we end up with (C.8). Finally, imposing ghost number three, (C.5), we conclude that

$$f(\{b_{-n}\}, \{\bar{b}_{-n}\}, c_0^+, [\text{matter}]) = c_0^+ g([\text{matter}]). \quad (\text{C.12})$$

Thus we have showed that a state obeying (C.3, C.4, C.5) is necessarily matter-ghost factorized

$$|B\rangle = |B\rangle^{\text{matter}} \otimes |B_{gh}\rangle, \quad (\text{C.13})$$

and the ghost factor $|B_{gh}\rangle \equiv |B_{bc}\rangle$ is the usual boundary state of the bc BCFT

$$(b_n - \bar{b}_{-n})|B_{bc}\rangle = 0 \quad (\text{C.14})$$

$$(c_n + \bar{c}_{-n})|B_{bc}\rangle = 0. \quad (\text{C.15})$$

From the total gluing conditions (C.4) we then find that $|B_\Psi\rangle^{\text{matter}}$ obeys the standard gluing conditions of the matter sector

$$(L_n^{\text{matter}} - \bar{L}_{-n}^{\text{matter}})|B\rangle^{\text{matter}} = 0, \quad (\text{C.16})$$

and from this it is easy to check that that $|B\rangle$ is also BRST invariant

$$(Q + \bar{Q})|B\rangle = 0. \quad (\text{C.17})$$

Few other universal gluing conditions follow from here. Let's look at the anomalous gluing of the ghost current

$$j_{gh}(z) = - :bc: (z) = \sum_n j_n z^{-n-1},$$

which, using the gluing conditions (C.14, C.15) reads

$$(j_n + \bar{j}_{-n} - 3\delta_{n0})|B\rangle = 0, \quad (\text{C.18})$$

Notice that, (C.5), we have $Q_{gh} = j_0 + \bar{j}_0$. From here it also follows that the BRST current

$$j_{BRST}(z) = \sum_n Q_n z^{-n-1}$$

glues non-anomalously at the boundary. Indeed we have that

$$Q_n = [Q, j_n], \quad (C.19)$$

and thus from the ghost current gluing condition it follows that

$$(Q_n + \bar{Q}_{-n})|B\rangle = 0. \quad (C.20)$$

C.2 Normalization of the ghost boundary state

Here we fix the normalization of the ghost boundary state from modular invariance. Consider a cylinder of circumference L and height T , $C_{L,T}$. We put BCFT₀ boundary conditions on the lower and upper boundary of $C_{L,T}$. We are interested in computing the partition function

$$\langle 1 \rangle_{C_{L,T}}.$$

In string theory, this partition function is identically vanishing because the zero modes of the b, c ghosts are not soaked up. On the cylinder there is a zero mode for c associated to the constant conformal Killing vector (CKV) for rotation of the cylinder around its axis. There is also a zero mode for b , associated with the constant holomorphic quadratic differential (HQD) which changes the length of the base circumference. Because both the CKV and the HQD are constant we have that

$$\langle b(w)c(w') \rangle_{C_{L,T}} \equiv Z_{L,T} \quad (C.21)$$

$$\partial_w \partial_{w'} \langle b(w)c(w') \rangle_{C_{L,T}} = 0, \quad (C.22)$$

see figure (C.1). To compute $Z_{L,T}$ we proceed as follows. We first consider a cylinder of height $T = \pi$ and circumference $L = 2\pi t$. Every $Z_{L,T}$ can be reduced to $Z_{2\pi t, \pi}$ by simple scaling keeping track of the weights of the insertions

$$\begin{aligned} Z_{L,T} = \langle bc \rangle_{C_{L,T}} &= \langle f \circ b f \circ c \rangle_{C_{2\pi t, \pi}} \Big|_{t=\frac{L}{2T}} = \frac{\pi}{T} Z_{2\pi t, \pi} \Big|_{t=\frac{L}{2T}} \\ f(w) &= \frac{\pi w}{T} \end{aligned} \quad (C.23)$$

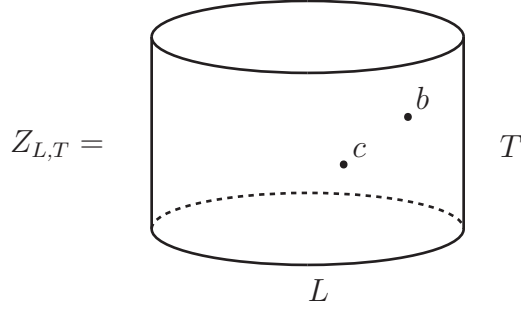


Figure C.1: The partition function $Z_{L,T}$ is given by the path integral on a cylinder $C_{L,T}$, with insertion of b and c . The position of the insertions is inessential as only the constant zero modes in the expansion of b and c gives nonvanishing contribution to the path integral.

$Z_{2\pi t, \pi}$ is just the one loop open string vacuum amplitude (before integration in moduli space)²⁰

$$\text{Tr}_{H_{\text{open}}} [(-1)^F e^{-2\pi t L_0} b_0 c_0] = \langle bc \rangle_{C_{2\pi t, \pi}} = Z_{2\pi t, \pi}. \quad (\text{C.24})$$

This follows from the fact that the cylinder is obtained by identifying the edges of a canonical open string strip of height π and length $2\pi t$. Such a strip is the image of the half annulus in the UHP (defined by $1 \leq |z| \leq e^{2\pi t}$ and $\Im z \geq 0$), obtained from the map

$$w = \ln z.$$

The UHP zero modes b_0, c_0 are mapped to vertical line integrals in the w coordinate ($w = y + ix$, doubling trick is used)

$$w \circ b_0 = \oint_0 \frac{dz}{2\pi i} z [w \circ b(z)] = \frac{1}{2\pi} \int_0^{2\pi} dx b(y + ix) \rightarrow b(w) \quad (\text{C.25})$$

$$w \circ c_0 = \oint_0 \frac{dz}{2\pi i} \frac{1}{z^2} [w \circ c(z)] = \frac{1}{2\pi} \int_0^{2\pi} dx c(y + ix) \rightarrow c(w). \quad (\text{C.26})$$

Lastly, we have ‘averaged’ the integrals because the correlator only gets contribution from the cylinder zero modes which are constant in the w coordinate. This establishes (C.24), see figure C.2

We can equivalently compute $Z_{2\pi t, \pi}$ by evolving the boundary state $|B\rangle$ with the closed

²⁰We define the fermion number

$$F \equiv Q_{gh} - \frac{3}{2} = \int_0^{2\pi i} \frac{dw}{2\pi i} j_{gh}(w),$$

as the zero mode of the ghost current in the canonical strip frame (with doubling trick understood). F is anti-hermitian and thus $(-1)^F$ is hermitian. The projector in Siegel gauge $b_0 c_0$ is anti-hermitian and the trace we are computing is imaginary.

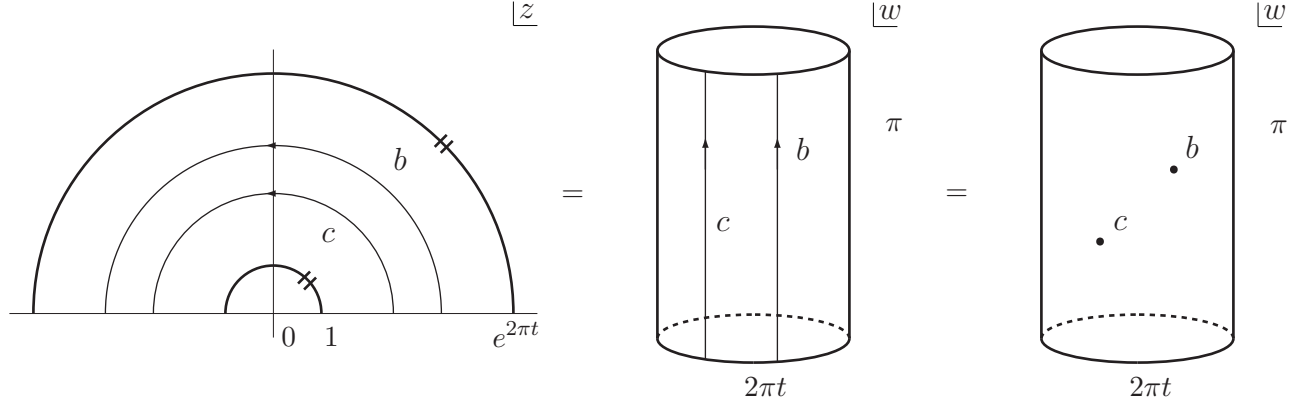


Figure C.2: Open string trace as a path integral on the cylinder, as stated in eq.(C.24). The inner and outer semicircles in the z coordinate are identified by the $\text{Tr}[(-1)^F(\dots)]$.

string propagator and contracting with the *BPZ dual* $\langle B|$. Proceeding similarly to the open string picture, we can write (using the ghost gluing conditions)

$$\begin{aligned}
\langle B|e^{-\frac{\pi}{t}(L_0+\bar{L}_0)}(b_0+\bar{b}_0)(c_0-\bar{c}_0)|B\rangle &= 4\langle B|e^{-\frac{\pi}{t}(L_0+\bar{L}_0)}b_0c_0|B\rangle \\
&= -4i \langle bc\rangle_{C_{2\pi,\frac{\pi}{t}}} \\
&= -4i Z_{2\pi,\frac{\pi}{t}}.
\end{aligned} \tag{C.27}$$

As illustrated in figure C.3, this is easily obtained by mapping the annulus $1 \leq |z| \leq e^{\frac{\pi}{t}}$ to the cylinder $C_{2\pi,\frac{\pi}{t}}$ with

$$w = i \left(\frac{\pi}{t} - \log z \right),$$

and replacing the resulting *horizontal* line integrals with a local insertion using again that the HQD and the CKV for b and c are constant on the cylinder ($w = y + ix$)

$$b_0 \rightarrow \frac{1}{2\pi} \int_0^{2\pi} dy b(y + ix) \rightarrow b(w) \tag{C.28}$$

$$c_0 \rightarrow -\frac{i}{2\pi} \int_0^{2\pi} dy c(y + ix) \rightarrow -i c(w). \tag{C.29}$$

Now we use the scaling property (C.23)

$$Z_{2\pi,\frac{\pi}{t}} = t Z_{2\pi t,\pi}, \tag{C.30}$$

to get

$$\langle B|e^{-\frac{\pi}{t}(L_0+\bar{L}_0)}(b_0+\bar{b}_0)(c_0-\bar{c}_0)|B\rangle = -4it \text{Tr}_{H_{open}} \left[(-1)^F e^{-2\pi t L_0} b_0 c_0 \right]. \tag{C.31}$$

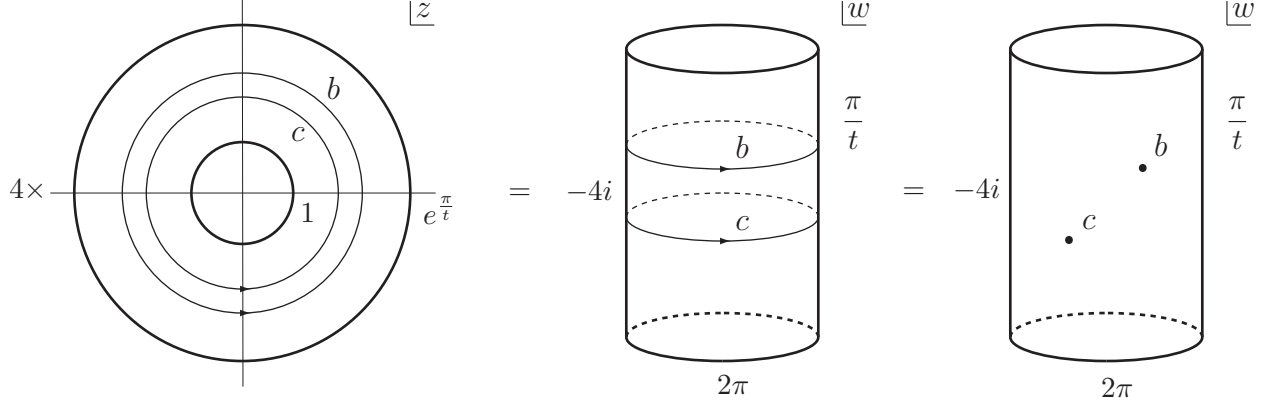


Figure C.3: Graphical representation of eq. (C.27).

This is an equality between two *real* quantities. We can use the above equation to determine the overall normalization of the ghost boundary state. A generic boundary state for an open string background is given by

$$|B\rangle = |B_{bc}\rangle \otimes |B_{\text{matter}}\rangle, \quad (\text{C.32})$$

where $|B_{\text{matter}}\rangle$ is the matter boundary state ($c = 26$) obeying Cardy condition

$$\langle B_{\text{matter}} | e^{-\frac{\pi}{t}(L_0^{\text{matter}} + \bar{L}_0^{\text{matter}} - \frac{c}{12})} | B_{\text{matter}} \rangle = \text{Tr}_{H_{\text{open}}^{\text{matter}}} \left[e^{-2\pi t(L_0 - \frac{c}{24})} \right], \quad (\text{C.33})$$

and

$$|B_{bc}\rangle = \mathcal{N}_{gh} (c_0 + \bar{c}_0) \exp \left(- \sum_{n=1}^{\infty} \bar{b}_{-n} c_{-n} + b_{-n} \bar{c}_{-n} \right) c_1 \bar{c}_1 |0\rangle_{SL(2,C)} \quad (\text{C.34})$$

$$\langle B_{bc} | = -\mathcal{N}_{gh} \langle 0 | c_{-1} \bar{c}_{-1} \exp \left(\sum_{n=1}^{\infty} \bar{b}_n c_n + b_n \bar{c}_n \right) (c_0 + \bar{c}_0). \quad (\text{C.35})$$

are the ghost boundary state and its *BPZ dual* whose normalization we want to determine. Taking the ghost part of (C.31) and assuming (C.33) we get a Cardy-like condition for the *bc*-BCFT

$$\langle B_{bc} | e^{-\frac{\pi}{t}(L_0 + \bar{L}_0 + \frac{26}{12})} (b_0 + \bar{b}_0)(c_0 - \bar{c}_0) | B_{bc} \rangle = -4it \text{Tr}_{H_{\text{open}}^{\text{ghost}}} \left[(-1)^F e^{-2\pi t(L_0 + \frac{26}{24})} b_0 c_0 \right]. \quad (\text{C.36})$$

Computing both sides with (C.34, C.35) we have

$$\begin{aligned} \langle B_{bc} | e^{-\frac{\pi}{t}(L_0 + \bar{L}_0 + \frac{26}{12})} (b_0 + \bar{b}_0)(c_0 - \bar{c}_0) | B_{bc} \rangle &= -2\mathcal{N}_{gh}^2 \eta^2 \left(\frac{i}{t} \right) \langle 0 | c_{-1} \bar{c}_{-1} (c_0 - \bar{c}_0)(c_0 + \bar{c}_0) c_1 \bar{c}_1 | 0 \rangle \\ &= -2\mathcal{N}_{gh}^2 \eta^2 \left(\frac{i}{t} \right) (-2) = 4\mathcal{N}_{gh}^2 \eta^2 \left(\frac{i}{t} \right), \end{aligned} \quad (\text{C.37})$$

where

$$\eta(it) = e^{-\frac{\pi t}{12}} \prod_{n=1}^{\infty} (1 - e^{-2\pi n t})$$

is the Dedekind η -function and we normalize the ghost BPZ-inner product as²¹

$$\langle 0 | c_{-1} c_0 c_1 \bar{c}_{-1} \bar{c}_0 \bar{c}_1 | 0 \rangle \equiv 1. \quad (\text{C.38})$$

Computing the open string trace and using the usual modular property of the Dedekind function

$$\sqrt{t} \eta(it) = \eta\left(\frac{i}{t}\right), \quad (\text{C.39})$$

we find

$$-4it \operatorname{Tr}_{H_{open}^{\text{ghost}}} \left[(-1)^F e^{-2\pi t (L_0 + \frac{26}{24})} b_0 c_0 \right] = 4t \eta^2(it) = 4\eta^2\left(\frac{i}{t}\right). \quad (\text{C.40})$$

This gives

$$\mathcal{N}_{gh}^2 = 1. \quad (\text{C.41})$$

Notice how the scaling law (C.30) accounts for the modular transformation (C.39).

We can fix the sign in \mathcal{N}_{gh} by asking

$$\langle B_{bc} | (c_0 - \bar{c}_0) | c\bar{c} \rangle = \langle (c_0 - \bar{c}_0) c\bar{c}(0) \rangle_{\text{disk}}, \quad (\text{C.42})$$

the rhs can be computed by expressing $(c_0 - \bar{c}_0)$ as a contour integral, mapping the disk to the upper half plane and using the doubling trick, by normalizing the basic ghost correlator in the usual way

$$\langle c(z_1) c(z_2) c(z_3) \rangle = z_{12} z_{13} z_{23}. \quad (\text{C.43})$$

This gives

$$\langle (c_0 - \bar{c}_0) c\bar{c}(0) \rangle_{\text{disk}} = -2, \quad (\text{C.44})$$

and the normalization of the ghost boundary state is thus given by

$$\mathcal{N}_{gh} = 1. \quad (\text{C.45})$$

²¹In the closed string Hilbert space, Hermitian and BPZ conjugation differ by an overall factor of i . In our conventions (slightly different from [46])

$$\text{BPZ}(|0\rangle) \equiv \langle 0| = i {}_{hc}\langle 0| \equiv i(|0\rangle)^\dagger.$$

The basic hermitian inner product is thus given by

$${}_{hc}\langle 0 | c_{-1} c_0 c_1 \bar{c}_{-1} \bar{c}_0 \bar{c}_1 | 0 \rangle \equiv -i,$$

and it agrees with textbooks conventions, [47], to which we adhere in this paper.

D Some more lumps

Here we collect the gauge invariant data of few more lump solutions. All data have been obtained in the $(L, 3L)$ scheme up to $L = 12$.

- Single lump at $R = \sqrt{3}$

This is the same solution of MSZ [34] but in the $(L, 3L)$ scheme.

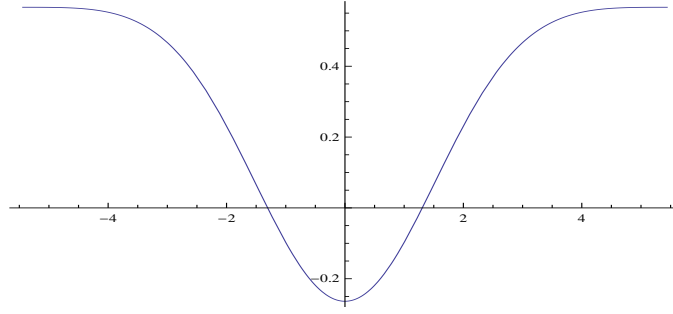


Figure D.1: Open string tachyon profile of the MSZ single-lump solution obtained at $R = \sqrt{3}$ and level $L = (12, 36)$.

L	Action	E_0	E_1	E_2	E_3	E_4	E_5	E_6	D
1	1.32002	1.23951	0.74368	—	—	—	—	—	1.23951
2	1.09428	1.09094	0.830804	1.03277	—	—	—	—	-1.01353
3	1.06053	1.06017	0.905713	1.08758	1.36793	—	—	—	-1.11078
4	1.03572	1.04623	0.918393	0.931471	1.4122	—	—	—	-0.752479
5	1.02936	1.03948	0.94075	0.933722	0.678169	—	—	—	-0.779229
6	1.02141	1.02921	0.946315	0.995166	0.676601	2.06251	—	—	-0.945165
7	1.01868	1.02668	0.956761	0.996584	1.11184	2.11211	—	—	-0.959492
8	1.01454	1.02301	0.959784	0.977037	1.12839	-0.327725	—	—	-0.909528
9	1.01351	1.02171	0.965702	0.976881	0.859033	-0.350675	3.66745	—	-0.913363
10	1.01108	1.01787	0.967666	0.993958	0.860958	1.98806	3.81063	—	-0.963774
11	1.01052	1.01708	0.971569	0.993933	1.04829	2.01551	-4.09339	—	-0.966875
12	1.00893	1.01549	0.972933	0.98699	1.05428	-0.0353736	-4.26896	8.53484	-0.945928
Exp.	1	1	1	1	1	1	1	1	-1

- Single lump at $R = 2\sqrt{3}$

This is a single lump centered at $x = \pi R$. Notice how this reflects into alternating signs for the E_n invariants.

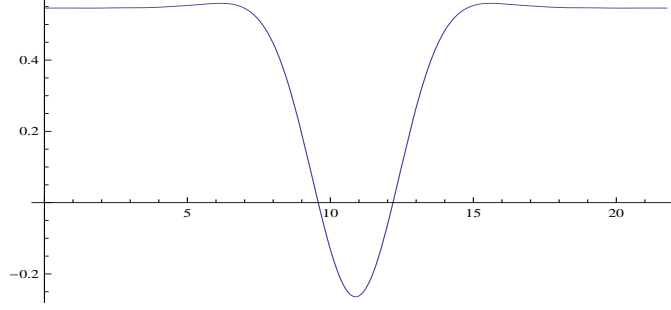


Figure D.2: Open string tachyon profile of a single-lump solution obtained at $R = 2\sqrt{3}$ and level $L = (12, 36)$.

L	Action	D	E_0	E_1	E_2	E_3	E_4	E_5
1	1.84419	1.70138	1.70138	- 0.72894	0.747337	- 0.723963	-	-
2	1.20098	- 0.467964	1.30548	- 0.77845	0.846539	- 0.91968	0.949864	-
3	1.13151	- 0.90771	1.25135	- 0.880111	0.906433	- 0.928996	1.08763	- 1.21329
4	1.05813	- 0.644043	1.1658	- 0.893358	0.922018	- 0.945655	0.92909	- 1.25847
5	1.05079	- 0.677223	1.15751	- 0.926427	0.941257	- 0.963228	0.933319	- 0.854891
6	1.02895	- 0.851623	1.11426	- 0.931712	0.947518	- 0.970614	0.993906	- 0.85989
7	1.02724	- 0.875745	1.11142	- 0.945795	0.957258	- 0.970967	0.997313	- 1.03384
8	1.01773	- 0.84127	1.09036	- 0.948533	0.960557	- 0.974792	0.976872	- 1.04075
9	1.01724	- 0.847968	1.08895	- 0.957215	0.966076	- 0.979099	0.977358	- 0.949906
10	1.01217	- 0.909139	1.07312	- 0.958953	0.968161	- 0.981546	0.993873	- 0.952527
11	1.01204	- 0.915063	1.07228	- 0.964312	0.971903	- 0.981671	0.994434	- 1.00923
12	1.00897	- 0.897553	1.06302	- 0.965506	0.973333	- 0.983331	0.98709	- 1.01181
Exp.	1	- 1	1	- 1	1	- 1	1	- 1

L	E_6	E_7	E_8	E_9	E_{10}	E_{11}	E_{12}
1	-	-	-	-	-	-	-
2	-	-	-	-	-	-	-
3	1.33609	-	-	-	-	-	-
4	1.38348	-	-	-	-	-	-
5	0.676515	- 1.67643	-	-	-	-	-
6	0.677062	- 1.72458	2.03206	-	-	-	-
7	1.1127	- 0.327417	2.12122	- 2.62668	-	-	-
8	1.12425	- 0.320042	- 0.314551	- 2.71207	-	-	-
9	0.859104	- 1.3691	- 0.352809	1.55456	3.69837	-	-
10	0.860785	- 1.3843	1.97233	1.61552	3.78564	-	-
11	1.04906	- 0.617564	2.01885	- 3.49068	- 4.12322	- 5.50346	-
12	1.05315	- 0.615319	- 0.0261646	- 3.56055	- 4.23092	- 5.60153	8.48854
Exp.	1	- 1	1	- 1	1	- 1	1

- Symmetric double lump at $R = 2\sqrt{3}$

This solution represents two D-branes at distance $a = \frac{1}{2}$, as defined in section 4.2. This double lump solution is just the single lump solution we obtained at $R = \sqrt{3}$, translated by $\pi\sqrt{3}$ and periodically extended to $R = 2\sqrt{3}$. This is clearly visible from the invariants which up to the alternating signs, are exactly the double of the single lump at $R = \sqrt{3}$.

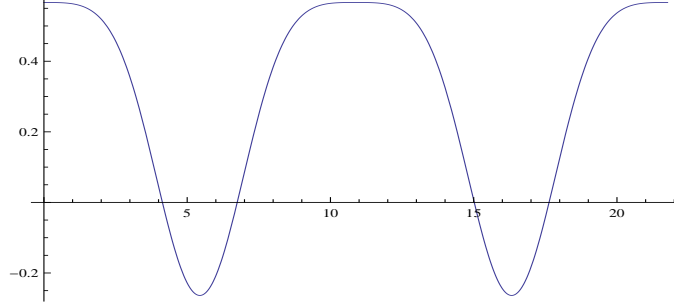


Figure D.3: Open string tachyon profile of a symmetric double-lump solution obtained at $R = 2\sqrt{3}$ and level $L = (12, 36)$.

L	Action	D	E_0	E_1	E_2	E_3	E_4	E_5
1	2.64005	2.47902	2.47902	0	-1.48736	-	-	-
2	2.18856	-2.02706	2.18187	0	-1.66161	0	2.06554	-
3	2.12107	-2.22155	2.12035	0	-1.81143	0	2.17516	0
4	2.07144	-1.50496	2.09245	0	-1.83679	0	1.86294	0
5	2.05871	-1.55846	2.07896	0	-1.8815	0	1.86744	0
6	2.04179	-1.89033	2.05843	0	-1.89263	0	1.99033	0
7	2.03736	-1.91898	2.05336	0	-1.91352	0	1.99317	0
8	2.02908	-1.81906	2.04602	0	-1.91957	0	1.95407	0
9	2.02702	-1.82673	2.04341	0	-1.9314	0	1.95376	0
10	2.02216	-1.92755	2.03574	0	-1.93533	0	1.98792	0
11	2.02103	-1.93375	2.03417	0	-1.94314	0	1.98787	0
12	2.01785	-1.89186	2.03098	0	-1.94587	0	1.97398	0
Expected	2	-2	2	0	-2	0	2	0

L	E_6	E_7	E_8	E_9	E_{10}	E_{11}	E_{12}
1	–	–	–	–	–	–	–
2	–	–	–	–	–	–	–
3	– 2.73586	–	–	–	–	–	–
4	– 2.82439	–	–	–	–	–	–
5	– 1.35634	0	–	–	–	–	–
6	– 1.3532	0	4.12501	–	–	–	–
7	– 2.22369	0	4.22422	0	–	–	–
8	– 2.25678	0	– 0.655451	0	–	–	–
9	– 1.71807	0	– 0.701349	0	– 7.3349	–	–
10	– 1.72192	0	3.97612	0	– 7.62125	–	–
11	– 2.09657	0	4.03102	0	8.18679	0	–
12	– 2.10857	0	– 0.0707473	0	8.53792	0	17.0697
Expected	– 2	0	2	0	– 2	0	2

References

- [1] E. Witten, “Noncommutative Geometry And String Field Theory,” Nucl. Phys. B **268**, 253 (1986).
- [2] W. Taylor and B. Zwiebach, “D-branes, tachyons, and string field theory,” arXiv:hep-th/0311017.
- [3] E. Fuchs and M. Kroyter, “Analytical Solutions of Open String Field Theory,” arXiv:0807.4722 [hep-th].
- [4] M. Kiermaier, Y. Okawa and B. Zwiebach, “The boundary state from open string fields,” arXiv:0810.1737 [hep-th].
- [5] I. Ellwood, “The Closed string tadpole in open string field theory,” JHEP **0808** (2008) 063 [arXiv:0804.1131 [hep-th]].
- [6] T. Kawano, I. Kishimoto and T. Takahashi, “Gauge Invariant Overlaps for Classical Solutions in Open String Field Theory,” Nucl. Phys. B **803** (2008) 135 [arXiv:0804.1541 [hep-th]].
- [7] N. Ishibashi, “The Boundary and Crosscap States in Conformal Field Theories,” Mod. Phys. Lett. A **4** (1989) 251.
- [8] A. Sen, “Universality of the tachyon potential,” JHEP **9912** (1999) 027 [hep-th/9911116].
- [9] L. Rastelli and B. Zwiebach, “Tachyon potentials, star products and universality,” JHEP **0109** (2001) 038 [hep-th/0006240].

- [10] D. Takahashi, “The boundary state for a class of analytic solutions in open string field theory,” JHEP **1111** (2011) 054 [arXiv:1110.1443 [hep-th]].
- [11] A. Rajaraman and M. Rozali, “D-branes in linear dilaton backgrounds,” JHEP **9912** (1999) 005 [hep-th/9909017].
- [12] M. Gaberdiel, “Boundary conformal field theory and D-branes”, Lectures given at the TMR network school on Nonperturbative methods in low dimensional integrable models, Budapest, 15-21 July 2003. <http://www.phys.ethz.ch/mrg/lectures2.pdf>
- [13] M. Kiermaier and Y. Okawa, “Exact marginality in open string field theory: A General framework,” JHEP **0911** (2009) 041 [arXiv:0707.4472 [hep-th]].
- [14] E. Fuchs, M. Kroyter and R. Potting, “Marginal deformations in string field theory,” JHEP **0709** (2007) 101 [arXiv:0704.2222 [hep-th]].
- [15] L. Bonora, C. Maccaferri and D. D. Tolla, “Relevant Deformations in Open String Field Theory: a Simple Solution for Lumps,” JHEP **1111** (2011) 107 [arXiv:1009.4158 [hep-th]].
- [16] M. Murata and M. Schnabl, “Multibrane Solutions in Open String Field Theory,” arXiv:1112.0591 [hep-th].
- [17] M. Schnabl, “Comments on marginal deformations in open string field theory,” Phys. Lett. B **654** (2007) 194 [hep-th/0701248 [HEP-TH]].
- [18] M. Kiermaier, Y. Okawa, L. Rastelli and B. Zwiebach, “Analytic solutions for marginal deformations in open string field theory,” JHEP **0801** (2008) 028 [hep-th/0701249 [HEP-TH]].
- [19] T. Erler, “Marginal Solutions for the Superstring,” JHEP **0707** (2007) 050 [arXiv:0704.0930 [hep-th]].
- [20] M. Kiermaier, Y. Okawa and P. Soler, “Solutions from boundary condition changing operators in open string field theory,” JHEP **1103** (2011) 122 [arXiv:1009.6185 [hep-th]].
- [21] M. Schnabl, “Analytic solution for tachyon condensation in open string field theory,” Adv. Theor. Math. Phys. **10** (2006) 433 [arXiv:hep-th/0511286].
- [22] Y. Okawa, “Comments on Schnabl’s analytic solution for tachyon condensation in Witten’s open string field theory,” JHEP **0604** (2006) 055 [arXiv:hep-th/0603159].

- [23] T. Erler, “Split string formalism and the closed string vacuum,” JHEP **0705** (2007) 083 [arXiv:hep-th/0611200].
- [24] T. Erler, “Split string formalism and the closed string vacuum. II,” JHEP **0705** (2007) 084 [arXiv:hep-th/0612050].
- [25] M. Schnabl, “Algebraic solutions in Open String Field Theory - a lightning review,” arXiv:1004.4858 [hep-th].
- [26] I. Kishimoto, “Comments on gauge invariant overlaps for marginal solutions in open string field theory,” Prog. Theor. Phys. **120** (2008) 875 [arXiv:0808.0355 [hep-th]].
- [27] T. Noumi and Y. Okawa, “Solutions from boundary condition changing operators in open superstring field theory,” JHEP **1112** (2011) 034 [arXiv:1108.5317 [hep-th]].
- [28] T. Erler and C. Maccaferri, “The Phantom Term in Open String Field Theory,” JHEP **1206** (2012) 084 [arXiv:1201.5122 [hep-th]].
- [29] T. Erler and C. Maccaferri, “Connecting Solutions in Open String Field Theory with Singular Gauge Transformations,” JHEP **1204** (2012) 107 [arXiv:1201.5119 [hep-th]].
- [30] A. Sen, “Tachyon dynamics in open string theory,” Int. J. Mod. Phys. A **20** (2005) 5513 [hep-th/0410103].
- [31] F. Larsen, A. Naqvi and S. Terashima, “Rolling tachyons and decaying branes,” JHEP **0302** (2003) 039 [hep-th/0212248].
- [32] I. Ellwood, “Rolling to the tachyon vacuum in string field theory,” JHEP **0712** (2007) 028 [arXiv:0705.0013 [hep-th]].
- [33] S. Hellerman and M. Schnabl, “Light-like tachyon condensation in Open String Field Theory,” arXiv:0803.1184 [hep-th].
- [34] N. Moeller, A. Sen, B. Zwiebach, “D-branes as tachyon lumps in string field theory,” JHEP **0008** (2000) 039. [hep-th/0005036].
- [35] N. Moeller, “Codimension two lump solutions in string field theory and tachyonic theories,” [hep-th/0008101].
- [36] M. Beccaria, “D0-brane tension in string field theory,” JHEP **0509** (2005) 021 [hep-th/0508090].
- [37] M. Kudrna, M. Schnabl, *to appear*

- [38] M. Kudrna, T. Masuda, Y. Okawa, M. Schnabl, K. Yoshida. “Gauge– invariant observables and marginal deformations in open string field theory”, arXiv:1207.3335 [hep-th].
- [39] T. Erler, “Exotic Universal Solutions in Cubic Superstring Field Theory,” JHEP **1104** (2011) 107 [arXiv:1009.1865 [hep-th]].
- [40] A. Sen and B. Zwiebach, “Large marginal deformations in string field theory,” JHEP **0010** (2000) 009 [hep-th/0007153].
- [41] J. L. Karczmarek and M. Longton, “SFT on separated D-branes and D-brane translation,” arXiv:1203.3805 [hep-th].
- [42] H. G. Kausch, “Curiosities at $c = -2$,” hep-th/9510149.
- [43] D. Gaiotto and L. Rastelli, “A Paradigm of open / closed duality: Liouville D-branes and the Kontsevich model,” JHEP **0507** (2005) 053 [hep-th/0312196].
- [44] M. Schnabl, “Wedge states in string field theory,” JHEP **0301** (2003) 004 [hep-th/0201095].
- [45] A. LeClair, M. E. Peskin and C. R. Preitschopf, “String Field Theory on the Conformal Plane. 1. Kinematical Principles,” Nucl. Phys. B **317** (1989) 411.
- [46] B. Zwiebach, “Closed string field theory: Quantum action and the B-V master equation,” Nucl. Phys. B **390** (1993) 33 [hep-th/9206084].
- [47] J. Polchinski, “String theory. Vol. 1: An introduction to the bosonic string,” Cambridge, UK: Univ. Pr. (1998) 402 p